DISCRETE-TIME CONVOLUTION

In today's class

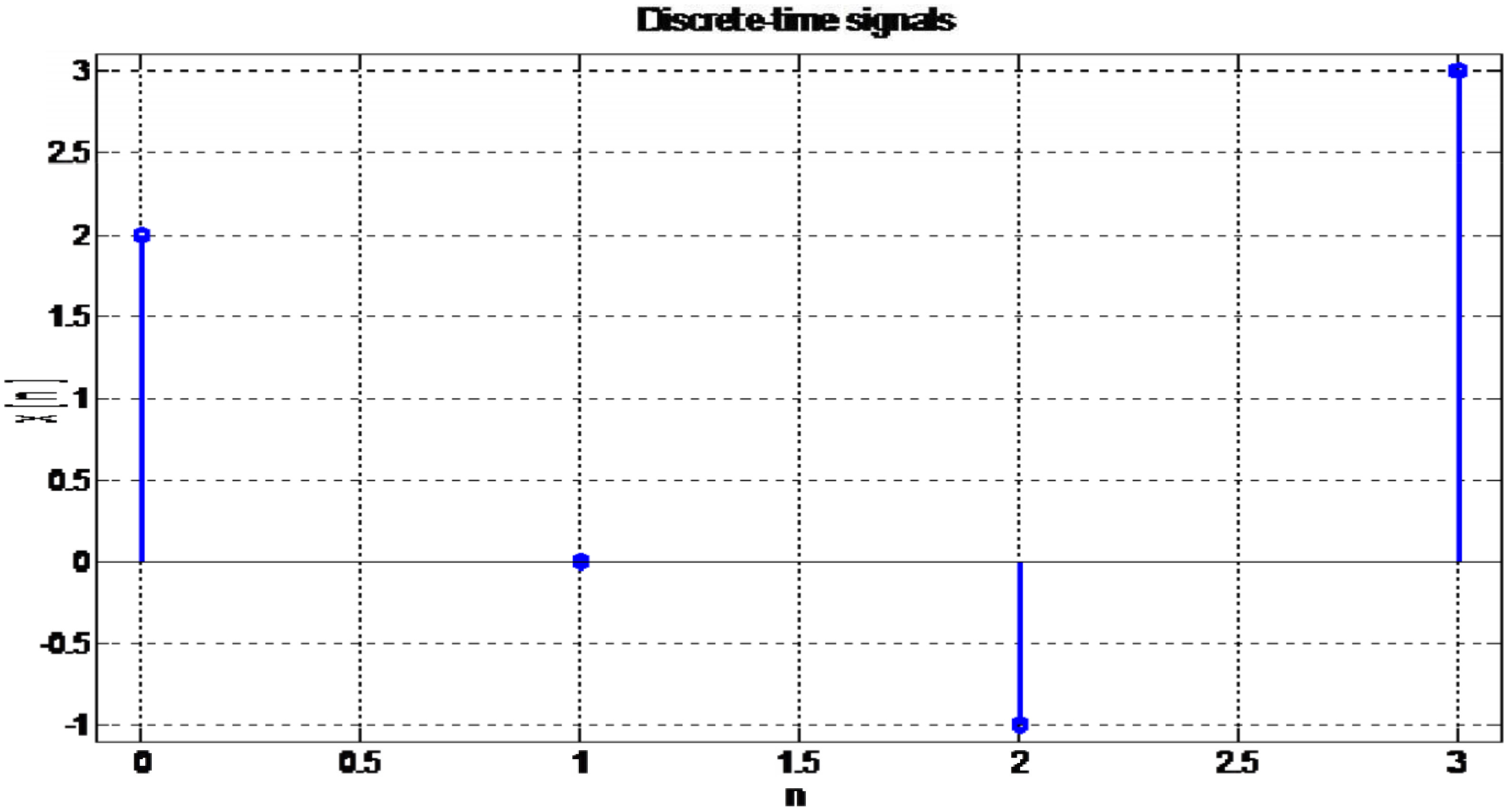
RESPONSE OF LTI SYSTEMS

- RESOLUTION OF INPUT INTO IMPULSES
  - DISCRETE-TIME INPUTS
- THE CONVOLUTION SUM
  - CHARACTERIZATION OF LTI SYSTEMS BY IMPULSE RESPONSE
  - PROPERTIES OF CONVOLUTION

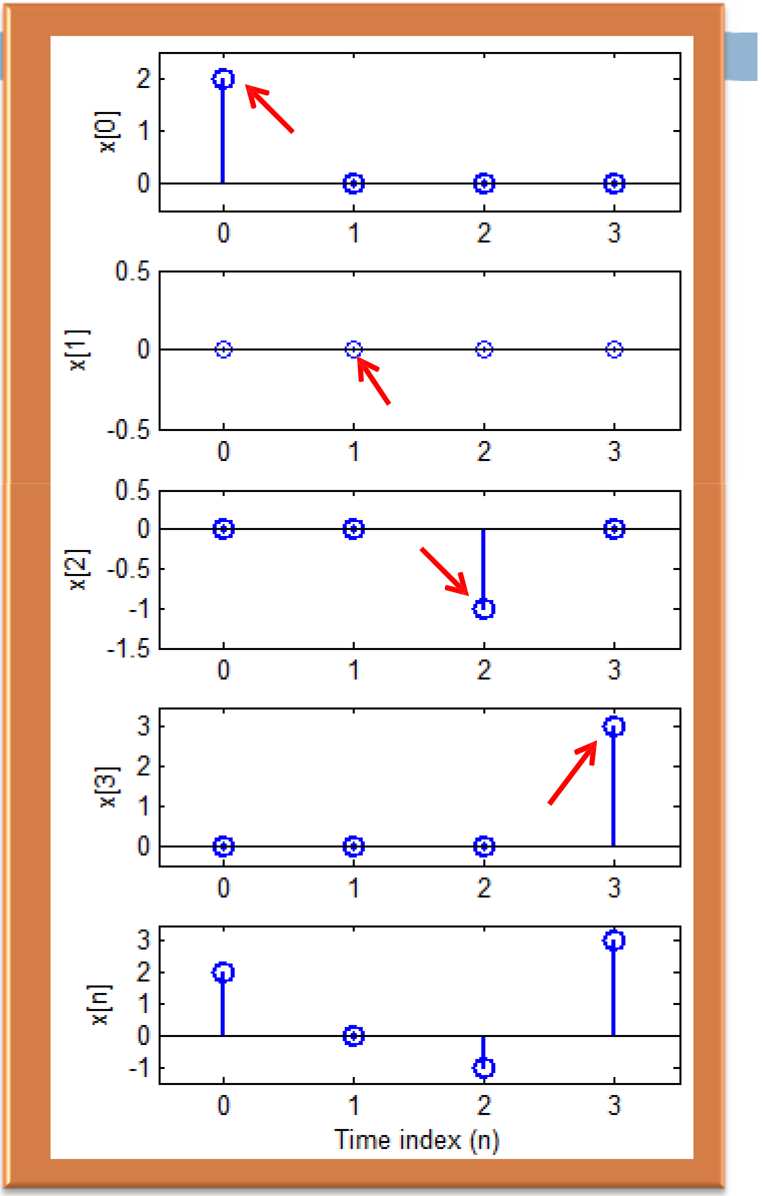
Discrete-time signals

☐ A discrete-time signal is a set of numbers

☐ x=[2 0 -1 3]



Resolution of a DT Signal into pulses



x = [2 0 -1   3]

Impulses at n = 0, 1, 2, and 3 with amplitudes

x[0] = 2, x[1] = 0, x[2] = -1, x[3] = 3

This can be written as,

$$x[n] = 2\delta[n] - \delta[n-2] + 3\delta[n-3]$$

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]$$

$$x[n] = \sum_{k=0}^{K-1} x[k]\delta[n-k]$$

K is the length of x

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

For infinite pulses

Example 1: Resolve the following discrete-time signals into impulses

$$x[n] = 2\ \underset{\uparrow}{4}\ 0\ 3 \qquad\qquad r[n] = 2\ 4\ \underset{\uparrow}{0}\ 3$$

Impulses occur at n = -1, 0, 1, 2 with amplitudes x[-1] = 2, x[0] = 4, x[1] = 0, x[2] = 3

$$x[n] = \sum_{k=-1}^{2} x[m]\delta[n-k]$$

$$= x[-1]\delta[n-(-1)] + x[0]\delta[n-0] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$x[n] = 2\delta[n+1)] + 4\delta[n] + 3\delta[n-2]$$

<u>Follow the same procedure for r[n]</u>

Characterization of LTI systems

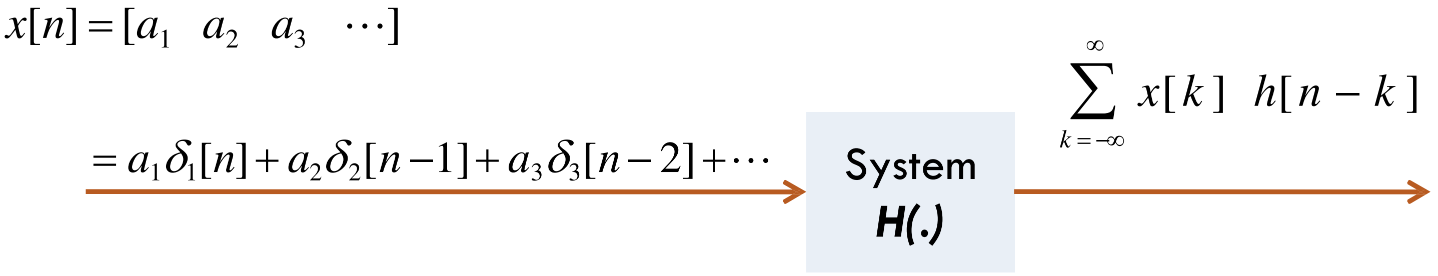
- LTI systems can be characterized in two ways
- Using Difference equations
  - Relationship between discrete-time inputs and discrete-time outputs
  - Also called Input-Output equations

$$y[n] = x[n] + \tfrac{3}{4}\, x[n-1] + 2x[n-5]$$

$$y[n] = x[n] + \tfrac{3}{4}\, x[n-1] + 2x[n-5] - \tfrac{1}{11}\, y[n-1] + \tfrac{5}{7}\, y[n-4]$$

Characterization of LTI systems

- Pulse response
  - System’s response to an impulse
  - Decompose the input signal vector into weighted-time-shifted impulses
  - Find the output of the system as the sum of its impulse response



$$x[n] = [a_1 \quad a_2 \quad a_3 \quad \cdots]$$

$$= a_1\delta_1[n] + a_2\delta_2[n-1] + a_3\delta_3[n-2] + \cdots$$

System

H(.)

$$\sum_{k=-\infty}^{\infty} x[k] \; h[n-k]$$

Convolution

- Convolution is the process by which an input interacts with an LTI system to produce an output

- Convolution between of an input signal x[n] with a system having impulse response h[n] is given as,

$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k] \; h[n-k]$$

where $*$ denotes the convolution

Convolution sum

We have already established that we can resolve the discrete-time input as weighted, time-shifted impulses

Lets generalize this

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Now, we apply this signal to an LTI system 'H' to get an output 'y'

$$y[n] = H\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \; h[n-k]$$

where h[n] is the response of the system H to each impulse

$$\boxed{y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \; h[n-k]}$$

Ways to find D.T. Convolution

Three ways to perform digital convolution

- S^{3}A graphical method
  - Scale, Shift, Stack, Add stack

- FSMA/Table method
  - Flip, Shift, Multiply, Add

- Analytical method

FSMA/Table method

Steps to follow:

Step 1	List the index ‘k’ covering a sufficient range
Step 2	List the input x[k]
Step 3	Obtain the reversed sequence h[-k], and align the rightmost element of h[n-k] to the leftmost element of x[k]
Step 4	Cross-multiply and sum the nonzero overlap terms to produce y[n]
Step 5	Slide h[n-k] to the right by one position
Step 6	Repeat step 4; stop if all the output values are zero or if required.

Example 2: Find the convolution of the two sequences x[n] and h[n] given by,

$$x[k] = [\underset{\uparrow}{3}\ 1\ \ 2] \qquad h[k] = [\underset{\uparrow}{3}\ 2\ 1]$$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

Hint: The value of k starts from **(– length of h + 1)** and continues till **(length of h + length of x – 1)**

Here ***k*** starts from **-3 + 1 = -2** and continues till **3 + 3 – 1 = 5**

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

$y[0] = 3 \times 3 = 9$

$y[1] = 3 \times 2 + 3 \times 1 = 9$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

y[2] = 3 x 1 + 1 x 2 + 2 x 3= 11

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

y[2] = 3 x 1 + 1 x 2 + 2 x 3= 11

y[3] = 1 x 1 + 2 x 2 = 5

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

y[2] = 3 x 1 + 1 x 2 + 2 x 3= 11

y[3] = 1 x 1 + 2 x 2 = 5

y[4] = 2 x 1 = 2

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y[0] = 3 x 3 = 9

y[1] = 3 x 2 + 3 x 1 = 9

y[2] = 3 x 1 + 1 x 2 + 2 x 3= 11

y[3] = 1 x 1 + 2 x 2 = 5

y[4] = 2 x 1 = 2

y[5] = 0 (no overlap)

$$\boxed{y[n]=\{9 \quad 9 \quad 11 \quad 5 \quad 2 \quad 0\}}$$

Example 3: Find the convolution of the two sequences x[n] and h[n] represented by,

$$x[n] = [\underset{\uparrow}{1}\ \ 2\ \ 4] \qquad h[n] = [\underset{\uparrow}{1}\ \ 1\ \ 1\ \ 1\ \ 1]$$

Example 4: Find the convolution of the two sequences x[n] and h[n] represented by,

$$x[n] = \{2\ \ \underset{\uparrow}{1}\ \ -2\ \ \ 3\ -4\} \qquad h[n] = [\underset{\uparrow}{3}\ \ 1\ \ 2\ \ 1\ \ 4]$$

Analytical method

In this method the Convolution sum can be found out by Analytical, meaning, using the formula for the Convolution sum

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Example 5: Find the output y[n] of a Linear, Time-Invariant system having an impulse response h[n], when an input signal x[n] is applied to it

$$h[n] = a^n u[n], \quad |a| < 1 \qquad\qquad x[n] = u[n]$$

By definition of Convolution sum, the output y[n] is given as

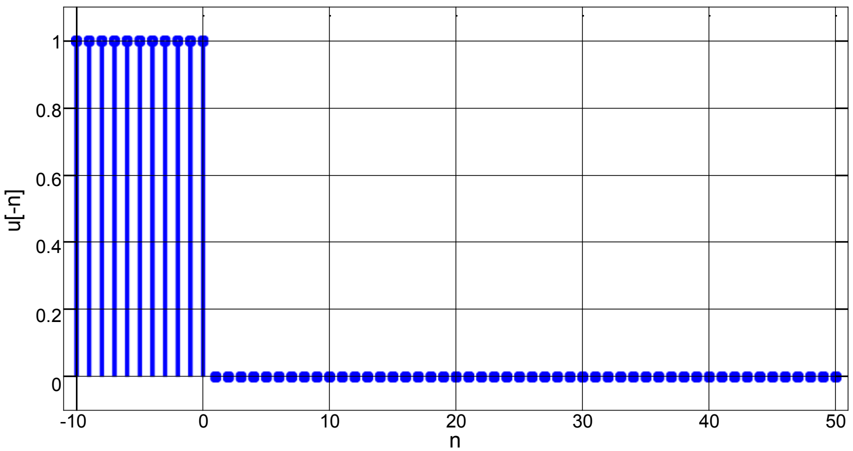
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\; h[n-k]$$

$$= \sum_{k=0}^{\infty} (1)\; a^n u[n-k] = \sum_{k=0}^{\infty} a^n u[n-k]$$

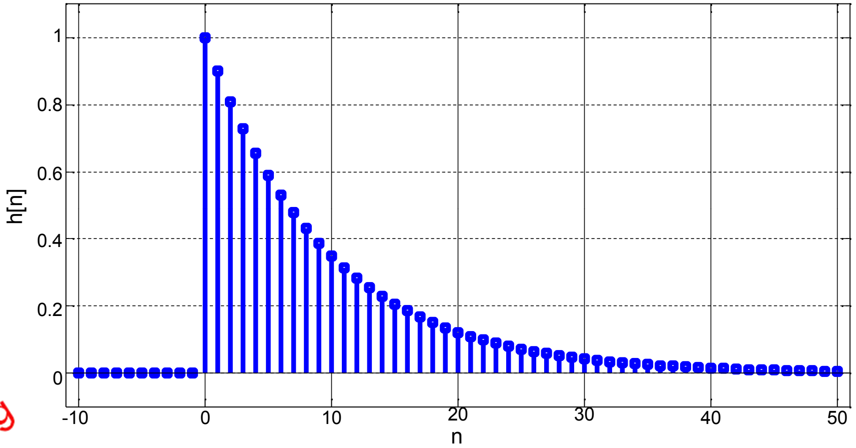
$$= a^0 + a^1 + a^2 + a^3 + \cdots$$

$$\boxed{y[n] = \frac{1}{1-a}}$$

Graphical representation is given on next slide x[-n]

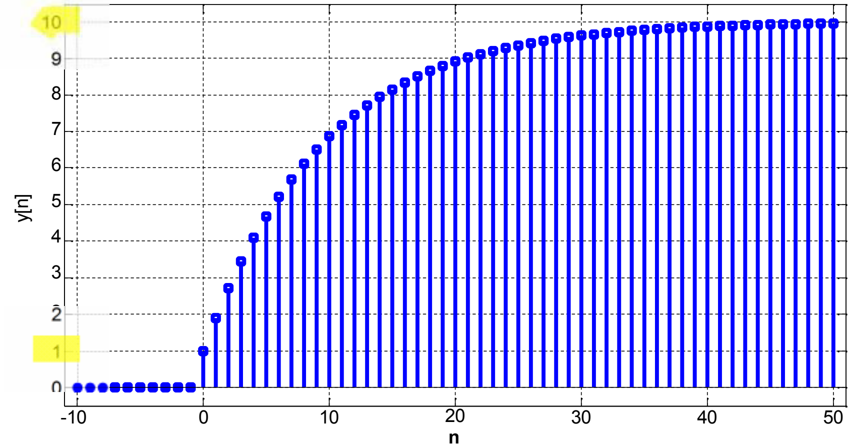


The System Pulse Response



The result of convolution

Note the difference in scaling of figure 3



Properties of Convolution

Commutative…

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

Associative…

$$\{x_1[n] * x_2[n]\} * x_3[n] = x_1[n] * \{x_2[n] * x_3[n]\}$$

Distributive…

$$\{x_1[n] + x_2[n]\} * x_3[n] = x_1[n]x_3[n] + x_2[n]x_3[n]$$