DISCRETE-TIME CONVOLUTION
In today’s class

RESPONSE OF LTI SYSTEMS

- RESOLUTION OF INPUT INTO IMPULSES
  - DISCRETE-TIME INPUTS

- THE CONVOLUTION SUM
  - CHARACTERIZATION OF LTI SYSTEMS BY IMPULSE RESPONSE
  - PROPERTIES OF CONVOLUTION
Discrete-time signals

- A discrete-time signal is a set of numbers
- $x=[2 \ 0 \ -1 \ 3]$
Resolution of a DT Signal into pulses

\[ x = [2 \quad 0 \quad -1 \quad 3] \]

Impulses at \( n = 0, 1, 2, \) and \( 3 \) with amplitudes
\[ x[0] = 2, \quad x[1] = 0, \quad x[2] = -1, \quad x[3] = 3 \]

This can be written as,
\[
x[n] = 2\delta[n] - \delta[n-2] + 3\delta[n-3]
\]
\[
x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]
\]
\[
x[n] = \sum_{k=0}^{K-1} x[k] \delta[n-k] \quad \text{K is the length of } x
\]
\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \quad \text{For infinite pulses}
\]
Example 1: Resolve the following discrete-time signals into impulses

<table>
<thead>
<tr>
<th>n</th>
<th>x[n]</th>
<th>r[n]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 4 0 3</td>
<td>2 4 0 3</td>
</tr>
</tbody>
</table>

Impulses occur at n = -1, 0, 1, 2 with amplitudes x[-1] = 2, x[0] = 4, x[1] = 0, x[2] = 3

\[
x[n] = \sum_{k=-1}^{2} x[m] \delta[n - k]
\]

\[
= x[-1] \delta[n - (-1)] + x[0] \delta[n - 0] + x[1] \delta[n - 1] + x[2] \delta[n - 2]
\]

\[
x[n] = 2 \delta[n + 1] + 4 \delta[n] + 3 \delta[n - 2]
\]

Follow the same procedure for r[n]
Characterization of LTI systems

- LTI systems can be characterized in two ways
- Using Difference equations
  - Relationship between discrete-time inputs and discrete-time outputs
  - Also called Input-Output equations

\[ y[n] = x[n] + \frac{3}{4} x[n-1] + 2x[n-5] \]

\[ y[n] = x[n] + \frac{3}{4} x[n-1] + 2x[n-5] - \frac{1}{11} y[n-1] + \frac{5}{7} y[n-4] \]
Characterization of LTI systems

- Pulse response
  - System’s response to an impulse
  - Decompose the input signal vector into weighted-time-shifted impulses
  - Find the output of the system as the sum of its impulse response

\[ x[n] = [a_1, a_2, a_3, \ldots] \]

\[ = a_1 \delta_1[n] + a_2 \delta_2[n-1] + a_3 \delta_3[n-2] + \cdots \]

\[ \sum_{k=-\infty}^{\infty} x[k] \ h[n-k] \]
Convolution

- Convolution is the process by which an input interacts with an LTI system to produce an output.

- Convolution between of an input signal $x[n]$ with a system having impulse response $h[n]$ is given as,

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

where $*$ denotes the convolution.
Convolution sum

We have already established that we can resolve the discrete-time input as weighted, time-shifted impulses.

Let's generalize this

\[ \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \]

Now, we apply this signal to an LTI system ‘H’ to get an output ‘y’

\[ y[n] = H \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \right\} \]

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \]

\[ y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \]

where \( h[n] \) is the response of the system \( H \) to each impulse.
Ways to find D.T. Convolution

Three ways to perform digital convolution

- $S^3A$ graphical method
  - Scale, Shift, Stack, Add stack

- FSMA/Table method
  - Flip, Shift, Multiply, Add

- Analytical method
## FSMA/Table method

**Steps to follow:**

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>List the index ‘k’ covering a sufficient range</td>
</tr>
<tr>
<td>Step 2</td>
<td>List the input $x[k]$</td>
</tr>
<tr>
<td>Step 3</td>
<td>Obtain the reversed sequence $h[-k]$, and align the rightmost element of $h[n-k]$ to the leftmost element of $x[k]$</td>
</tr>
<tr>
<td>Step 4</td>
<td>Cross-multiply and sum the nonzero overlap terms to produce $y[n]$</td>
</tr>
<tr>
<td>Step 5</td>
<td>Slide $h[n-k]$ to the right by one position</td>
</tr>
<tr>
<td>Step 6</td>
<td>Repeat step 4; stop if all the output values are zero or if required.</td>
</tr>
</tbody>
</table>
**Example 2:** Find the convolution of the two sequences $x[n]$ and $h[n]$ given by,

$$x[k] = [3 \ 1 \ 2] \quad h[k] = [3 \ 2 \ 1]$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[k]$</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[1-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[2-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[3-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[4-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h[5-k]$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Hint:** The value of $k$ starts from ($-\text{length of } h + 1$) and continues till ($\text{length of } h + \text{length of } x - 1$)

Here $k$ starts from $-3 + 1 = -2$ and continues till $3 + 3 - 1 = 5$
\[ k: \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

<table>
<thead>
<tr>
<th>( x[k] )</th>
<th>3</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h[-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( h[1-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( h[2-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( h[3-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( h[4-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( h[5-k] )</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ y[0] = 3 \times 3 = 9 \]
<table>
<thead>
<tr>
<th>k</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x[k]</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[1-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[2-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[3-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[4-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[5-k]</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y[0] = 3 \times 3 = 9 \]
\[ y[1] = 3 \times 2 + 3 \times 1 = 9 \]
<table>
<thead>
<tr>
<th>k:</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x[k]:</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[1-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[2-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[3-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[4-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[5-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ y[0] = 3 \times 3 = 9 \]
\[ y[1] = 3 \times 2 + 3 \times 1 = 9 \]
\[ y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11 \]
<table>
<thead>
<tr>
<th>k</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x[k]:</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[1-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[2-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[3-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[4-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h[5-k]:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
y[0] = 3 \times 3 = 9
\]
\[
y[1] = 3 \times 2 + 3 \times 1 = 9
\]
\[
y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11
\]
\[
y[3] = 1 \times 1 + 2 \times 2 = 5
\]
\[
\begin{array}{cccccccc}
k: & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
x[k]: & 3 & 1 & 2 & & & & & \\
h[-k]: & 1 & 2 & 3 & & & & & \\
h[1-k]: & 1 & 2 & 3 & & & & & \\
h[2-k]: & 1 & 2 & 3 & & & & & \\
h[3-k]: & 1 & 2 & 3 & & & & & \\
h[4-k]: & 1 & 2 & 3 & & & & & \\
h[5-k]: & 1 & 2 & 3 & & & & & \\
\end{array}
\]

\[
y[0] = 3 \times 3 = 9 \\
y[1] = 3 \times 2 + 3 \times 1 = 9 \\
y[2] = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11 \\
y[3] = 1 \times 1 + 2 \times 2 = 5 \\
y[4] = 2 \times 1 = 2 \\
\]
\[
y[n] = \{9, 9, 11, 5, 2, 0\}
\]
Example 3: Find the convolution of the two sequences $x[n]$ and $h[n]$ represented by,

$$
x[n] = [1 \ 2 \ 4]
\uparrow
h[n] = [1 \ 1 \ 1 \ 1 \ 1]
\uparrow
$$
Example 4: Find the convolution of the two sequences $x[n]$ and $h[n]$ represented by,

$$x[n] = \{2 \ 1 \ -2 \ 3 \ -4\} \quad h[n] = [3 \ 1 \ 2 \ 1 \ 4\]$$
Analytical method

In this method the Convolution sum can be found out by Analytical, meaning, using the formula for the Convolution sum

\[ y[n] = x[n] * h[n] \]

\[ = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]
Example 5: Find the output $y[n]$ of a Linear, Time-Invariant system having an impulse response $h[n]$, when an input signal $x[n]$ is applied to it

$$h[n] = a^n u[n], \quad |a| < 1 \quad x[n] = u[n]$$

By definition of Convolution sum, the output $y[n]$ is given as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \ h[n-k]$$

$$= \sum_{k=0}^{\infty} (1) \ a^n u[n-k] = \sum_{k=0}^{\infty} a^n u[n-k]$$

$$= a^0 + a^1 + a^2 + a^3 + \cdots$$

$$y[n] = \frac{1}{1-a}$$

Graphical representation is given on next slide
The System Pulse Response

The result of convolution

Note the difference in scaling of figure 3
Properties of Convolution

Commutative...

\[ x_1[n] \ast x_2[n] = x_2[n] \ast x_1[n] \]

Associative...

\[ \{ x_1[n] \ast x_2[n] \} \ast x_3[n] = x_1[n] \ast \{ x_2[n] \ast x_3[n] \} \]

Distributive...

\[ \{ x_1[n] + x_2[n] \} \ast x_3[n] = x_1[n]x_3[n] + x_2[n]x_3[n] \]