Abstract
This paper extends Sraffa’s model by eliminating two of its main hypotheses, i.e. the uniformity of the rate of profit and the zero rate of accumulation. Its aim is to elaborate a common formalization to equilibrium and disequilibrium and to sketch the dynamics of a Classical disequilibrium model from a non-gravitational perspective.

We build a model with given methods and exogenous viable accumulation rates. Two possible rules of distribution of the non-accumulated part of production among capitalists are examined, giving rise to two distinct models of Classical 'reproduction prices'. In both cases the rates of profit admit a physical interpretation and are positively correlated to the relative rates of accumulation. Model 1 behaves like Torrens's reproduction model when all profits are accumulated. Sraffa’s solution is obtained in model 2 when the rates of accumulation and profit are both uniform.

The general structure of the dynamics of a simplified two-sector economy is made of three modules, dealing successively with physical reproduction, the plans for accumulation and their implementation. The analysis of physical reproduction brings to light the changes in the reproduction constraints from a period to the next. When the desired rates of accumulation are not compatible, we introduce a ‘short-side rule’ describing how the actual feasible rates of accumulation derive from the desired rates. The dynamics shows a great variety of trajectories.

Key words: accumulation, Classical theory, disequilibrium, dynamics, prices, profit, reproduction.

JEL classification: E11, E30, E32, O41
INTRODUCTION
This paper proposes a formalization which extends Sraffa’s theory of value to disequilibrium and non stationary states, and makes explicitly use of the concept of reproduction prices. Sraffa’s model is based on two explicit basic hypotheses: the means of production are reproduced identically and the rates of profit are equal. The extension we put forward does not suppose the uniformity of the rate of profit and takes into account a positive net accumulation. A model constructed on these bases can be called a model of “reproduction prices”.

In the Classical tradition, the most significant economic decisions are those relative to the levels of accumulation in the various sectors, while the unproductive capitalists’ expenditure is treated as a residue. The capitalists’ decisions, taken independently and motivated by the search for profit, are submitted to an objective social evaluation by means of the actual rates of profit. The equations of the classical theory of value reflect these decisions. For a given wage basket, stemming from the balance of power between capitalists and workers, the net product is entirely available to capitalists. Therefore, the actual rates of profit are endogenous distribution variables, and profits are divided between productive (accumulation) and unproductive expenditure.

In the first part of this paper we set out two models of classical “reproduction prices” derived from the extension of Sraffa’s theory of value to disequilibrium economies with positive rates of accumulation. The second part analyzes some dynamics of these models in a simple bisector economy.¹

I. REPRODUCTION PRICES
Section 1 defines a set of concepts and equations common to Sraffa and, basically, to all classical economists. Section 2 highlights the distinctive features of Sraffa’s theory of value. Section 3 introduces an “equation of circulation” which allows us to consider Sraffa’s model as a particular case of a general formalization, with two variants of which we study the respective properties. Section 4 distinguishes three types of equilibria: profitability equilibrium, physical reproduction equilibrium and full equilibrium. Section 5 shows that the rates of profit can be interpreted in physical terms; more specifically they are rooted in the physical characteristics of the reproduction system.

I.1. TOWARDS A GENERAL CLASSICAL FRAMEWORK

The Classicals considered that the produced quantities and the accumulation behaviour are given for the determination of current prices. All commodities can have a productive or an unproductive use. Examples of unproductive use are capitalist consumption and the payment of taxes to the government for the financing of unproductive labour. Following Sraffa, we do not make any explicit assumption on the use of the non accumulated part of production. By hypothesis, the whole production is sold and its unproductive use acts as a buffer. This is the way to express that the market is conceived as a social evaluation of production and investment decisions, whereas capitalist consumption aims at satisfying private needs but has no relevant social effect. The very idea of consumption as a part of the residual would be illegitimate from a neoclassical perspective in which individuals aim at maximizing intertemporal utility. But utility maximization is absent from the classical conception, which considers that the consumption behaviour is determined by social and historical conditions, with different norms for each social class.

For a given real wage advanced by capitalists, let us replace each unit of labour input by the corresponding wage basket, so that labour does not appear explicitly in the system. This basket may be consumed or not by the workers according to their choices. Let there be \( n \) single-product industries with constant returns, one technique and one producer in each sector:

\[
\forall \ i \quad x_{i1}, \ x_{i2}, \ldots, \ x_{in} \rightarrow y_i
\]

where \( x_{ij} \) is the quantity of commodity \( j \) necessary to produce the quantity \( y_i \) of commodity \( i \). The economy represented by the methods and quantities (1) is called the system \( C \) (\( C \) for concrete). Let us introduce the useful notion of a rate of surplus of a commodity (which in the productive system (1) is entirely available for profits). The rate of surplus \( s_i \) of commodity \( i \) over the present period is defined as the ratio between its net product in the economy and the whole investment in that commodity:

\[
\forall \ i \quad s_i = \frac{y_i}{x_{i1} + x_{i2} + \ldots + x_{in}} - 1
\]

The rates \( s_i \) depend on the proportions in which commodities are produced.

The rates of accumulation \( g_i \) decided by capitalists in each industry at date \( t \) determine the activity levels for the incoming period \( t \) and, therefore, the gross products at date \( t + 1 \). They also determine the amount available today for the unproductive use, as a difference
between production and invested quantities (denoted \( d_i \)). Note that the concept of a rate of accumulation concerns a sector, while that of a rate surplus is relative to a commodity.

The compatibility of the accumulation rates with the available production is expressed by the viability conditions:

\[
\forall i \quad (1 + g_1)x_{1i} + (1 + g_2)x_{2i} + \ldots + (1 + g_n)x_{ni} \leq y_i
\]

which is an equality for wholly accumulated commodities. Only the accumulation rates \( g_i \geq 0 \) will be retained.

Since the left-hand side of (3) is the denominator of the factor of surplus of commodity \( i \) in the next period \((1 + s^*_i)\), the viability conditions can be expressed as:

\[
\forall i \quad g_i \leq s^*_i
\]

The prices satisfy two types of conditions:

- They cover the costs and allow capitalists to obtain a profit on the capital invested. This condition is expressed by the “equations of production”:

\[
\forall i \quad (1 + r_i) (x_{1i}p_1 + x_{i2}p_2 + \ldots + x_{in}p_n) = y_i p_i
\]

Since the model retains the idea that the ‘historical aim’ of capitalism is capital accumulation, the rates of profit are calculated on the basis of replacement costs instead of historical costs.

- They ensure equality between receipts and expenditure in each sector. This condition is expressed by the equations we will call “equations of circulation”. While the equations of production are general, these equations are specific to each model and will be introduced in the following sections.

### I.2. SRAFFA’S REPRODUCTION PRICES

For exogenous and advanced wages, Sraffa’s model derives from the previous framework by introducing two hypotheses. Assume first that “no changes in output” (Preface of *Production of Commodities*) occur, so that all rates of accumulation \( g_i \) are zero; and, second, that the distribution of the value of surplus among sectors is proportional to the value of their capital because “the rate of profits must be uniform for all industries” (*ibid.*, § 4). At the stationary equilibrium, the relations

\[
\forall i \quad (1 + r) (x_{1i} p_1 + x_{i2} p_2 + \ldots + x_{in} p_n) = y_i p_i
\]

are the equations of production and, simultaneously, the equations of circulation: they express the equality in each sector between receipts \( y_i p \) and the sum of capital expenditure, means of production and wage-goods, \((x_{1i} p_1 + x_{i2} p_2 + \ldots + x_{in} p_n)\), and of unproductive expenditure by
capitalists, \( r (x_{i1} p_1 + x_{i2} p_2 + \ldots + x_{in} p_n) \). Solving (6) gives Sraffa’s reproduction prices and the uniform rate of profit, which only depend on technical conditions and distribution. The level of the rate of profit stands between the lowest and the highest rates of surplus. Let us change the physical proportions between sectors and call ‘homothetic system’ a system with a uniform rate of surplus, as in Sraffa’s “standard system”. Since the solution of (6) is independent of the proportions, the rate of profit is unchanged and equal to that uniform rate of surplus. This physical interpretation of the rate of profit is a meaningful property which will be extended to our models in section I.4.

According to another possible interpretation of Sraffa’s model, the concrete system admits positive and exogenous rates of accumulation. The equations of production (6) no longer coincide with the equations of circulation and determine the uniform rate of profit and the prices which are then prices of production. If these rates \( g_i \) are uniform, one obtains a Sraffian quasi-stationary equilibrium model. If they are different and viable (see condition (3)), the equations of circulation are read

\[
\forall i \quad (1 + g_i) (x_{i1} p_1 + \ldots + x_{in} p_n) + c_i r (x_{i1} p_1 + \ldots + x_{in} p_n) = y_i p_i
\]

where the non accumulated fractions \( c_i \) of the sectoral profits are determined endogenously. In some respect, the model has similarities with Marx’s schemes of extended reproduction: prices and the uniform rate of profit are fixed, like Marxian labour values and rates of profit. But these rates of accumulation cannot be maintained forever. In a truly disequilibrium reproduction model, the rates of profit and accumulation are unequal and change over time, as well as prices. Such an approach will be developed in the following sections.

I.3. REPRODUCTION PRICES IN DISEQUILIBRIUM

Let us discard the two hypotheses mentioned above which are at the basis of the Sraffian model. The equations of production expressed by (5) are maintained. Those expressing the equality between the value of gross production and expenditure depend now upon a hypothesis on the distribution of the non-accumulated part of production between sectors. In this study, we examine two possible rules: (i) each capitalist appropriates the value of his own production which is not demanded for accumulation (model 1); or, (ii) each capitalist accumulates the same fraction of his profits (model 2). Both hypotheses are compatible with the basic features we attribute to the classical approach and, within the common framework of section I.1, allow us to build two different models of “reproduction prices” in which the driving variables are the capitalists’ decisions of accumulation.
In both models, each capitalist knows *a priori* the supply of the good he produces and his demand of means of production (including the wage goods). The prices are such that production is entirely sold and, since they are displayed at the opening of the markets, the capitalists can calculate the *value* of their unproductive expenditure. Once the transactions on means of production are completed (by means of a centralized trading procedure), the remaining goods are available for unproductive use and exchanged.

I.3.1. First model

The first model assumes that the *value* of individual capitalist unproductive expenditure is equal to that of the non-accumulated part of his production. The corresponding equations of capitalists' expenditure are:

\[
\forall i \ (1 + g_i) (x_{i1} p_1 + x_{i2} p_2 + \ldots + x_{in} p_n) + v_i = y_i p_i
\]

where \(v_i = d_i p_i\) is the value of the \(i\)th capitalist’s unproductive expenditure. By replacing the physical production \((y_i)\) in the right-hand side of (7) by its productive and unproductive uses, namely \(((1 + g_1) x_{i1} + (1 + g_2) x_{i2} + \ldots + (1 + g_n) x_{in})\) and \(d_i\), one obtains

\[
\forall i \ (1 + g_i) \sum_{j \neq i} x_{ij} p_j = p_i \sum_{j \neq i} (1 + g_j) x_{ji}
\]

These \(n\) equations express that the value of the other commodities accumulated in each industry \(i\) equals the value of the amount of commodity \(i\) accumulated in the other industries. These equations reduce to \(n - 1\) and determine the \(n - 1\) relative prices. The noteworthy property of the relative prices is that they are only determined by the multilateral exchange of that part of production that will be invested. Prices only depend on the physical conditions of production and the rates of accumulation. Then, equations (5) determine the rates of profit. As these profit rates differ, the economy is in a profitability disequilibrium which, however, is compatible with the reproduction of the system during the period.

A change in the relative scale of sectors, though not affecting the equations of production, alters (8) and, therefore, prices and rates of profit. Thus, contrary to Sraffa’s system, the constancy of returns does not preclude that prices and rates of profit depend on the proportions between sectors. They also depend on the physical composition of the wage basket and invested capital.

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2 Note that in bisector economies \((i = 1, 2)\) a direct interpretation of (8) is that the value of capital bought by sector 1 to 2 equals that bought by sector 2 to 1. That is why it can be called ‘equation of exchange’.
Equations (8) show a positive relation between the relative price of a commodity and the relative factor of accumulation of the sector that produces it. In order to make the algebraic relationships between factors of accumulation and rates of profit more explicit, let us multiply the two members of (5) by \((1 + g_i)\) and, using (8), replace \((1 + g_i)\sum_{j \neq i} x_{ij} p_j\) by \(p_i \sum_{j \neq i} (1 + g_j) x_{ji}\). After elimination of \(p_i\), we obtain:

\[\forall i \quad (1 + r_i) [(1 + g_1) x_{i1} + \ldots + (1 + g_i) x_{ii} + \ldots + (1 + g_n) x_{in}] = (1 + g_i) y_i\]  
(9)

Relations (9) show that the rate of profit increases in the sector in relative expansion and decreases in the other (trade-off between the sector rates of profit). These conclusions are summarized by the scheme:

\[\forall i \quad (1 + g_i)/(1 + g_j) \rightarrow \{p_i/p_j, r_i, r_j\}\]
(10)

A parallel variation of the accumulation factors has no effect on prices and rates of profit.

The right-hand side of equality (9) represents the quantities of commodity \(i\) produced during the next period, whereas those invested at the beginning of that period appear into the bracket of the left-hand side. It thus turns out that the rate of profit in sector \(i\) over the present period is equal to the rate of surplus of the commodity \(i\) during the next period:

\[\forall i \quad r_i = s_i^+\]
(11)

The economic interpretation of (11) relies on the fact that the relative price is nothing but a relationship between accumulated quantities of commodities which generate the surplus of the following period.

Although it has been obtained as an extension of Sraffa’s system, our first model has non-Sraffian properties and can be considered as a formalization and generalization of Torrens insights on reproduction (see Chapter VI, section VI, of *Essay on the Production of Wealth*). Torrens’ basic hypothesis is that the whole product is accumulated. This means that the rates of accumulation are endogenous: at a given date, they are the solution of the relationships (3) considered as equalities. Since all profits are devoted to buy new inputs and allow the capitalist \(i\) to increase the vector of inputs by the rate \(\hat{g}_i\), it turns out that his rate of profit \(r_i\) is equal to \(\hat{g}_i\). In other terms, in Torrens’ model, the rate of profit is nothing but the rate of accumulation: \(r_i = \hat{g}_i\) and its physical interpretation is self-evident. Therefore, the equations of production coincide with those of circulation, like in Sraffa’s model but for a different

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3 There is no need of prices to compare two baskets of commodities when they are proportional.
reason. Since the whole product is accumulated, the viability condition is written $g_i = s_i^+$, hence (11). (Torrens suggests the equality $g_i = r_i$, but not equality (11)).

The main originality of Torrens’ model is to propose a determination of prices and rates of profit out of equilibrium. Our two models extend this possibility to economies with productive and unproductive capitalists’ expenditure (therefore, unlike Torrens’ model, the accumulation rates are exogenous in our models).

I.3.2. Second model

The second model assumes that the capitalists’ unproductive expenditure is an (endogenous) uniform proportion of their profits. Let $c$ be this proportion. The equations of circulation are written

$$\forall i \quad (1 + g_i) \left( x_{i1} p_1 + \ldots + x_{in} p_n \right) + c r_i \left( x_{i1} p_1 + \ldots + x_{in} p_n \right) = y_i p_i$$

(12)

where the first term on the left-hand side represents gross investment and the second is the value of unproductive expenditure of each producer, which depends on the proportion $c$, the sectoral rate of profit and prices. The $2n$ equations (5) and (12) determine the $2n$ unknowns, viz. the $n$ rates of profit $r_i$, the relative prices $p_i$ and the proportion $c$. As in the first model the profitability disequilibrium is compatible with the reproduction of the economy during the period.

The ratio between the value of gross production and that of capital in each sector is equal to the factor of profit $1 + r_i$ according to equations (5) and to $1 + g_i + cr_i$ according to equation (12). Hence the hypothesis on the distribution of the non accumulated part of production implies that, in each sector, the accumulation rate is equal to the product of the uniform rate of savings by the rate of profit

$$\forall i \quad g_i = (1 - c) r_i$$

(13)

Therefore, for each sector $i$ two of the three equations (5), (12) and (13) imply the third, so that one or another of the equations of production (5) or circulation (12) can be replaced by (13).

A method of resolution of the system is as follows. According to the equations (13), the sector rates of profit and of accumulation are in the same proportions:

$$\forall i, j \quad r_i/r_j = g_i/g_j$$

(14)

Knowing the rates of accumulation, equality (14) gives the structure of the rates of profit, then the equations of production (5) determine the rates of profit as well as the relative prices. Finally, the non accumulated fraction of profits is defined by (12) or (13). The solution only
depends on the value of capital and wages and a change in the activity levels does not affect equations (5), (12) and (13) so that the rates of profit and relative prices remain unchanged (those properties also hold for Sraffa’s model, but not for our model 1).

Equations (14) show a positive relation between the relative rates of profit and the relative rates of accumulation. More precisely, by taking into account the equations of production, we have the following scheme

\[
\forall i, j \quad g_i/g_j \rightarrow \{p_i/p_j, r_i, r_j\}
\]

(15)

\((+, +, -)\)

An increase in the level of accumulation rates, for a given relative structure, does not alter the rates of profit or prices, and only the capitalists’ non accumulated fraction of profits decreases. By contrast, when the relative rate of accumulation increases in an industry, the corresponding rate of profit and the relative price increase.

1.4. PHYSICAL INTERPRETATION OF THE RATES OF PROFIT

The classical school characteristically stresses the role of objective factors in economics. For instance, the idea that prices reflect the difficulty of production (be it measured or not by means of incorporated labour) and not the consumers’ preferences is shared by many Classicals. Similarly, the rate of profit is not merely the solution to a system of equations but is conceived as a magnitude that can be interpreted in objective terms. The idea is explicit in Marx’ theory: profits are the apparent form taken by surplus value, and the rate of profit depends on workers’ exploitation. As we saw, in Torrens’ theory, the rate of profit in a sector is nothing but its own rate of accumulation. More recently, Sraffa’s well-known interpretation of Ricardo’s Essay (1815) stresses the fact that, in the corn model, the rate of profit represents the ratio between two quantities of wheat: the part of the product going to the farmers and their advances. Sraffa’s notion of “standard system” generalizes the physical interpretation to multisector models (see section I.2).

One may wonder if the rates of profit defined in our models admit a similar interpretation. The idea we follow is to adapt Sraffa’s construction of the “standard system”. This system is built by applying multipliers to the concrete system: when proportions are adjusted so that the rate of surplus of all goods is uniform, the maximum rate of profit is this common rate of surplus. For our purpose, we discard that part of production unproductively used and isolate a subsystem named ‘core of accumulation’ which describes the production of capital by means of capital. The steps are as follows.
Consider the first model. Let the data be the concrete system (1) and viable rates of sectoral accumulation. We change the activity levels of the concrete system in order that the modified system produces these accumulated quantities exactly. For this we calculate the \( n \) coefficients \( k_i \) which express the accumulated fraction of the production in each sector:

\[
\forall i \quad k_i = \frac{x_i (1 + g_1) + x_2 (1 + g_2) + \ldots + x_n (1 + g_n)}{y_i}
\]

The transformed system, called system \( K \), is obtained by applying these coefficients to the initial concrete system:

\[
\forall i \quad k_i x_{i1}, k_i x_{i2}, \ldots, k_i x_{in} \rightarrow k_i y_i
\]

System \( K \) is the core of accumulation: it produces the quantities invested in the concrete system, and the produced quantities are totally accumulated. Unlike the concrete system, the rate of accumulation in each industry of system \( K \) is endogenous and is calculated so that the totality of the product is reinvested, as in Torrens’ model.

This core of accumulation has noticeable properties:

(i) Let us denote by the index \( K \) the variables in the core of accumulation. The accumulation rates \( (g_{Ki}) \) are the solution of the following equations:

\[
\forall i \quad (1 + g_{K1}) k_i x_{i1} + (1 + g_{K2}) k_i x_{i2} + \ldots + (1 + g_{K_n}) k_i x_{in} = k_i y_i
\]

The ‘Torrens’ property’ holds: since physical product is totally accumulated, the rates of accumulation are the rates of profit which in turn are the rates of surplus at the next period:

\[
\forall i \quad g_{Ki} = r_{Ki} = s_{Ki}^r
\]

(ii) From (16) and (18) we obtain:

\[
\forall i \quad (1 + g_{Ki}) k_i = (1 + g_i)
\]

Consequently, the effective system corresponding to the methods (1) when capitalists accumulate at rates \( g_i \) is the same as that obtained by the reinvestment of whole production at rates \( g_{Ki} \) in the system \( K \). Hence these two systems have the same rates of surplus at the next period: \( s_{Ki}^r = s_i^r \) and according to (11) and (19) we have:

\[
\forall i \quad r_i = g_{Ki}
\]

To sum up, we have shown that the rate of profit of sector \( i \) in the concrete system, which is also that of sector \( i \) in system \( K \), is equal, on the one hand, to the rate of surplus of good \( i \) in the concrete system at the next period (relation (11)) and, on the other, to the accumulation rate of sector \( i \) in the core (relation (21)). In the first model, the rates of profit can be given a physical interpretation: they are the physical rates of accumulation of the core of accumulation.
In the second model, a similar interpretation is obtained by a two-step procedure:

(i) First, let us apply adequately chosen multipliers to the concrete system $C$. These activity multipliers transform system $C$ into another productive system which we will call system $Q$. The extension of Sraffa’s idea consists in applying the multipliers $q_i$, ($i = 1, \ldots, n$) to $n$ sectors so that the rates of surplus of commodities are in the same proportion as the rates of accumulation. Moreover, in system $Q$, the rates of surplus are proportional to the rates of profit. The system $Q$ thus obtained only differs from the concrete system in its proportions, therefore both have the same rates of profit. On the other hand, all rates of profit cannot be higher than all rates of surplus (otherwise, the total profits would exceed the net product) nor can they be lower (for a symmetrical reason). By combining these arguments, we conclude that the rates of profit in all sectors are equal to the rates of surplus of the respective commodities in system $Q$.

(ii) Next, we extract a core (which we shall call system $K'$) out of system $Q$, by applying a procedure similar to that followed to extract $K$ out of $C$ but for an uniform rate of accumulation, nil to simplify. This amounts to define $K'$ as the system producing the means of production used in $Q$. Since $Q$ and $K'$ only differ in its proportions their solution is the same. The relationship between $K'$ and $Q$ is the same as between $K$ and $C$ mutatis mutandis. As the production of system $K'$ is wholly accumulated, the rates of profit of the concrete system are the accumulation rates in the core $K'$.

In conclusion, in both models the rates of profit can be interpreted in physical terms, by relying on the idea of production of capital by means of capital. The device, inspired by Torrens’ and Sraffa’s contributions, consists in building the core of accumulation.

### I.5. EQUILIBRIA

This section examines the conditions for equilibrium in both models. We distinguish three notions of equilibrium: (i) physical reproduction equilibrium, defined by the uniformity of the sectoral accumulation rates, (ii) profitability equilibrium, defined by the uniformity of the profit rates, and (iii) full equilibrium, defined by the simultaneous existence of both these equilibria.

#### I.5.1. Equilibria in the first model

In this model, the conditions for physical and profitability equilibria are different. The equilibrium of physical reproduction is generally compatible with the disparity of the rates of profit. In a given concrete system, the profitability equilibrium is only reached for a particular
vector of the accumulation factors, which is determined by equations (9) with \( r_i = r_j \) (all \( i, j \)). We know that these accumulation factors transform the current concrete system into a new system in which the surplus rates \( s_i^+ \) are equal to the current rates of profit \( r_i \). Since the previous rates of profit were uniform, the surplus rate of all commodities in the new system is uniform: this system is “homothetic”.

Full equilibrium is generally impossible in a given concrete economy, as equations (9) are incompatible with \( g_i = g_j \) and \( r_i = r_j \) (all \( i, j \)). Its implementation requires that the existing proportions between industries are those of a homothetic system. Then the surplus rates are equal and remain unchanged whilst the rates of accumulation are uniform. It follows that the rates of profit are also uniform, the quantity of each good available for capitalists’ unproductive use is a uniform proportion of production, and the value of unproductive expenditure of each capitalist is a uniform proportion of his profit. These proportions depend on the level of accumulation rates and full equilibrium prices are nothing but Sraffa’s prices.

In this model, like in Torrens’, full equilibrium only exists in a homothetic system (but, under Torrens’ hypothesis, the notions of full equilibrium and profitability equilibrium cannot be dissociated).

I.5.2. Equilibrium in the second model

According to (14), the equality \( g_i = g_j \neq 0 \) leads to \( r_i = r_j \). Therefore, apart from the stationary state, the physical reproduction equilibrium implies the profitability equilibrium, and full equilibrium is then achieved whatever the proportions between sectors. In the corresponding system \( Q \) all the rates of surplus, which are the rates of profit in the concrete system, are equal (as in Sraffa’s standard system). It is the same for the core of accumulation \( K’ \) which produces the means of production used in system \( Q \). This solution generalizes Sraffa’s system to a uniform non-zero rate of accumulation. In this case, the equality of the rates of profit becomes a result of the analysis and the equilibrium solution is the same in both models.

The stationary state must be distinguished from the general case. In this state, the rates of accumulation are nil and the whole profits unproductively used \((c = 1)\). In each industry, the equations of production and circulation become identical. The system of reproduction prices is then reduced to \( n \) equations, which cannot determine the \( n \) rates of profit and the \( n - 1 \) relative price. A solution requires other exogenous constraints, for instance the equality of the rates of profit. This possibility, retained in Sraffa’s formalization, is only one amongst many
alternatives. In the second model, a stationary state is a physical reproduction equilibrium, not necessarily a profitability equilibrium.

II. SOME DYNAMICS OF DISEQUILIBRIUM

The determination of prices and rates of profit out of equilibrium is a preliminary condition for the study of dynamics. Contemporary works related to dynamics in a classical perspective generally assume intersectoral mobility of capital and deal with the gravitation of market prices around normal prices. We explore a different approach: the dynamics rely essentially on the desired rates of accumulation, which may depend on past profits. Let us adapt the framework of part I but simplify it by considering a two-sector economy with two goods, wheat and iron. At a given date, the actual state of the economy is described by the size and the content of the product, which results from the past, and its use which depends on accumulation in each sector. The general structure of the dynamics is logically composed of three modules:

- The first deals with physical reproduction. It analyses the use of the quantities produced at a given date and, given the actual rates of accumulation (we shall see later how these are fixed), determines the production at the next date.
- The second treats the plans for accumulation.
- The third concerns the implementation of the investment projects: the desired rates of accumulation, which result from individual decisions, may be incompatible. A rule is applied to transform the desired rates into feasible actual rates.

The successive application of these three modules to the system at a certain date defines its state at the following date. It then becomes possible to study the short and long-term dynamics of the economy.

This part of the paper is organized as follows. Section 1 provides an analysis of physical reproduction, which brings to light the reproduction constraint on each good at a given date and the changes in these constraints from one period to the next. In this section, the rates of accumulation of the period are taken as parameters. In Section 2 the ‘short-side rule’ describes how the actual rates of accumulation derive from the desired rates. Section 3 combines the physical analyses of Sections 1 and 2 with the value analysis of the first part and studies some variants of dynamics according to the mode of formation of the desired rates. The dynamics is sensitive to the hypotheses retained on the agents’ behaviour. We have retained some simple rules, which however tend to eliminate some causes of disequilibrium.
noticed by the Classicals. The persistence of disequilibrium or the sub-optimality of equilibrium in spite of this optimistic choice is a meaningful result.

II.1. PHYSICAL REPRODUCTION

The unit period (year) is defined by the length of the production process. Period \( t - 1 \) begins at date \( t - 1 \) with the investment of the inputs \( x_j(t - 1) \) and ends at date \( t \) when the product \( y_j(t) \) is available. The production during year \( t - 1 \) is written:

\[
\begin{align*}
    x_1(t - 1) \text{ wheat} \oplus x_{12}(t - 1) \text{ iron} &\rightarrow y_1(t) \text{ wheat} \\
    x_2(t - 1) \text{ wheat} \oplus x_{22}(t - 1) \text{ iron} &\rightarrow y_2(t) \text{ iron}
\end{align*}
\]

Let the two factors of accumulation at date \( t \) be taken as parameters:

\[
G_i(t) = 1 + g_i(t) = \frac{x_j(t)}{x_j(t - 1)} = \frac{y_j(t + 1)}{y_j(t)}
\]

Accumulation in one sector imposes a constraint on the accumulation in the other sector. The aim of this section is to write down these reciprocal constraints at a given date and to study their modifications from one period to the next.

At a given date, the accumulation of good \( j \) is bounded from above by the available product

\[
x_{i,j}(t) + x_{j,i}(t) \leq y_j(t)
\]

hence the constraints on the factors of accumulation:

\[
\begin{align*}
    x_{11}(t - 1)G_1(t) + x_{21}(t - 1)G_2(t) &\leq y_1(t) \\
    x_{12}(t - 1)G_1(t) + x_{22}(t - 1)G_2(t) &\leq y_2(t)
\end{align*}
\]

In the positive orthant \((G_1, G_2)\) (Figure 1), these constraints define the OSTU zone of the feasible accumulation factors.
If \( G_1(t) \) and \( G_2(t) \) differ, the conditions (26) and (27) for the next date differ from those at the present date. Before studying the corresponding deformations of Figure 1, let us introduce some definitions.

Recall the definition of the rate of surplus of commodity \( j \) at date \( t + 1 \) (see (2)):

\[
s_j(t+1) = \frac{y_j(t+1)}{x_{1j}(t) + x_{2j}(t)} - 1
\]  

(28)

Since the invested quantities do not exceed those produced at date \( t \), we have (see (4)):

\[
s_j(t+1) \geq \frac{y_j(t+1)}{y_j(t)} - 1 = g_j(t)
\]  

(29)

This inequality becomes equality when the whole product \( j \) is accumulated. From formula (28), rewritten in terms of the factor of surplus \( S_1 = 1 + s_1 \) of good 1, there follows

\[
S_j(t+1) = \frac{1}{\frac{x_{1j}(t)}{y_j(t+1)} + \frac{x_{2j}(t)}{y_j(t+1)} \frac{y_2(t+1)}{y_j(t+1)}}
\]  

(30)

The formula shows that the rate of surplus of a commodity only depends on the relative activity levels. Therefore, from a date to the next, the rate of surplus of commodity 1 remains constant, increases or decreases according as the rate of accumulation in sector 1 is equal, higher or lower than in sector 2. Moreover, by eliminating \( y_2(t+1)/y_1(t+1) \) between
equality (30) for commodity 1 and the similar equality for commodity 2, it turns out that point \((S_1(t), S_2(t))\) is located on a decreasing hyperbola \((H)\) for any \(t\).

Summing up, the notions of factor of surplus and factor of accumulation are closely connected: the factor of surplus of commodity \(j\) is at least equal to the factor of accumulation in sector \(j\) decided at the previous date; more precisely, it depends of the last relative factors of accumulation in both sectors and, in turn, defines an upper limit to the next accumulation decisions.

Let \(G^*\) denote the \textit{maximum maximorum} factor of regular growth (or accumulation) of the economy. \(G^*\) is the inverse of the dominant eigenvalue of the input matrix associated with methods (22) and (23) and it is also the factor of surplus common to the two goods when the relative activity levels are defined by the Perron-Frobenius row-vector. Thus, the point \(T^*\) of coordinates \((G^*, G^*)\) belongs to the hyperbola \((H)\). In the economy (22)-(23), one of the commodities has a factor of surplus greater or equal to \(G^*\) and the other a factor smaller or equal to \(G^*\). For a given \(G = (G_1, G_2)\), Figure 1 itself usually changes from each date to the following. The deformations depend on the location of \(G:\)

(i) Diagonal ray and point \(T^*\)

Point \(G\) belongs to the diagonal ray \((\Delta)\) when the rates of accumulation are equal. The actual system (22)-(23) then reproduces itself homothetically. Consequently, neither the proportions between sectors nor the rates of surplus change. This is the only case in which Figure 1 is the same for \(t\) and \(t + 1\). Point \(T^*\) on the diagonal ray corresponds to the maximal rate of growth of the economy. It is reached if and only if, in the actual system, all commodities have the same rate of surplus and the entire surplus is accumulated. Otherwise, point \(T^*\) is situated outside the feasible zone \(OSTU\).

(ii) The \(STU\) frontier

Each point \(G\) of the frontier \(STU\) corresponds to the total accumulation of one commodity (and, at point \(T\) only, of both commodities). The frontier is composed of two segments \(TU\) and \(ST\), which we name the wheat-segment and the iron-segment respectively, according to the totally accumulated good. On \(ST\) and \(TU\), the rate of accumulation at date \(t\) of the sector that produces the entirely accumulated good is equal to the rate of surplus of this good at the date \(t + 1\) (condition (29) with equality).

(iii) Notable points on the frontier
At point $T$ (in reference to Torrens, 1821), and only at this point, the two commodities are totally accumulated in the actual system. As the rates of accumulation at date $t$ become the rates of surplus at date $t + 1$, point $T$ is located on the hyperbola ($H$): in particular, one of the components of $T$ is greater than $G^*$ and the other smaller.

Let us denote $l$ the commodity with the lower rate of surplus and $h$ the other commodity (we assume that in the economy (22)-(23), $l$ is iron and $h$ is wheat), and consider point $\alpha$ at the intersection of the frontier with $(\Delta)$. This point defines the higher rate of regular growth of the actual system, equal to the rate of surplus of the totally accumulated commodity. It can be shown that it is the rate of commodity $l$ (iron): if anyone coordinate of $T^*$ is reduced until the point belongs to the frontier, segment $l$ is reached.

Let $\beta$ be the point reached by reducing coordinate $h$ of $T^*$. At this point, by construction, the rate of accumulation of commodity $l$ is equal to $G^*$. Therefore, in the new productive system, the rate of surplus of commodity $l$ is equal to $G^*$, and this must also be the case for commodity $h$. To sum up, the productive system generated at $t + 1$ by the rates of accumulation represented by point $\beta$ is the homothetic system corresponding to the regular maximum growth path.

Points $S$ and $U$ correspond to situations wherein one sector invests the entire gross product of one commodity, which forbids its use in the other sector, causing the system to collapse. Points $\sigma$ and $\upsilon$ correspond to situations in which one sector completely absorbs the net product of a commodity, thus prohibiting the other one from having a positive rate of accumulation. If disaccumulation is excluded, the system enters a stationary state.

(iv) Points below the frontier

For points $G$ inside the $OSTU$ zone, the demand for accumulation is relatively weak and both goods are partly consumed. Any point $G$ belongs to a segment $OF$ where $F$ is on the frontier. What are the relationships between the systems generated by $G$ and by $F$? As the factors of accumulation are proportional, these systems only differ in size. Consequently, both have the same rate of surplus: for instance, the property of point $\beta$ to generate a homothetic system spreads to all points on segment $O\beta$.

(v) The regions

Let the $OSTU$ zone be divided into three regions bordered by the axes, the $STU$ frontier, the diagonal ray and the segment $O\beta$. This partition is illustrated by Figure 1, in which wheat is commodity $h$ and iron is $l$, and point $\beta$ is located above the diagonal. Within $OSTU$, let $A$
be the area delimited by the diagonal and the axis corresponding to commodity $h$ (in Figure 1, this area is under the diagonal), $B$ be the triangle $aO\beta$, and $C$ be the area between $O\beta$ and the axis of commodity $l$. We have already noticed that the rates of accumulation of the two commodities are equal when $G$ belongs to the diagonal, and that the rates of surplus will become equal in the new productive system when $G$ belongs to the segment $O\beta$. On the basis of these observations, the following conclusions are obtained:

- When the factors of accumulation are in region $A$ (triangle $OS\alpha$), the gap between the rates of surplus is amplified in the new productive system.
- When the factors of accumulation are in region $B$ (triangle $aO\beta$) the gap between the rates of surplus is reduced.
- When the factors of accumulation are in region $C$ (quadrilateral $O\beta TU$), commodities $h$ and $l$ will be inverted in the new productive system. In the next period, the region $A$ will be above the diagonal.

From one period to the other, frontier $STU$ and the regions move according to the variations of the rates of surplus, as derived from the factors of accumulation. The detailed study of these movements is complex but it is possible since, whatever the region where factors of accumulation are located, we know their effect on rates of surplus. If $G$ belongs to segment $O\beta$, the new constraints intersect at $T^*$. If $G$ belongs to ($\Delta$), the constraints do not vary (case of homothetic growth).

II.2. FROM DESIRED TO ACTUAL RATES

II.2.1. The short-side rule

Since the accumulation decisions of ‘the’ farmer and of ‘the’ metallurgist (the behaviours are aggregated by industry) expressed by the desired (or planned) rates are taken independently, their compatibility is not guaranteed. Let $g_i^d$ be the desired rate of accumulation of the capitalist of sector $i$ ($i = 1, 2$). This decision expresses that the capitalist $i$ devotes a part of his own product to accumulation and demands the other product, that he wishes to buy. The desired rates become effective if they are compatible with the quantities produced, the non-accumulated part being used unproductively by the capitalists. But the desired rates may also be incompatible, namely when one of the goods, or even both, is in excess demand with regard to the available quantities. Then, the corresponding market will be considered to be in a crisis and at least one of these rates must be revised: the desired and the actual investments will differ. The short-side rule adopted here assumes that each capitalist $i$ keeps for himself the quantity of good $i$ that he would like to accumulate and only brings to the market the
amount he wishes to sell. The other capitalist \( j \) demands this good either partly (case of excess supply) or totally (in the exceptional case of a balanced market or when the capitalist \( j \) is rationed on good \( i \)). In the economy (22)-(23), the quantities exchanged for accumulation are formally defined by:

\[
\begin{align*}
\min \left( y_1(t) - x_{11}(t-1) G_1^d, x_{21}(t-1) G_2^d \right) & \quad \text{for wheat} \\
\min \left( y_2(t) - x_{22}(t-1) G_2^d, x_{12}(t-1) G_1^d \right) & \quad \text{for iron.}
\end{align*}
\]

(31) (32)

II.2.2. Visualization of the rule

Figure 2 represents the desired factors of accumulation \( G^d = (G_1^d, G_2^d) \) and the effective factors \( G = (G_1, G_2) \). The short-side rule assigns to each point \( G^d \) an effective point \( G \) in the OSTU zone. As the planned factor \( G_i^d \) of capitalist \( i \) does not exceed \( y_i/x_{ii} \) (which corresponds to the reinvestment in sector \( i \) of the totality of its product), point \( G^d \) is located in the rectangle \( ODEF \) of Figure 2. The two lines

\[
\begin{align*}
x_{11} G_1 + x_{21} G_2 &= y_1 \\
x_{12} G_1 + x_{22} G_2 &= y_2
\end{align*}
\]

(33) (34)

expressing the constraints on maximum demand for wheat and iron are the borders of four zones.
- In zone 1 (OSTU) where the desired accumulation rates are feasible, we have \( G = G^d \).

- In zone 2 (triangle UTF), there are simultaneously an excess supply of iron (no capitalist is constrained in this market) and an excess demand for wheat. According to the short-side rule, the farmer keeps a provision of wheat for himself and, being not constrained in any market, fulfils his accumulation plan \( G_1 = G_1^d \). The weight of adjustment falls entirely on the metallurgist who must reduce his planned accumulation because wheat is scarce \( (G_2 < G_2^d) \). The effective point \( G \) is obtained by lowering the second component of \( G_2^d \) until the wheat constraint is reached.

- In zone 3 (triangle STD), the conclusions are symmetric with those in zone 2: \( G \) is obtained by decreasing the first component \( G_1^d \) until the iron segment is reached.

- Zone 4 (TDEF) corresponds to an excess demand on both markets because the desired rates of accumulation are high. As the farmer keeps the quantity of wheat he wants, his accumulation constraint is determined by the amount of iron left over by the metallurgist, that is, \( G_1 = (y_2 - x_22 G_2^d) / x_12 \). According to this formula, all the points \( G^d \) of zone 4 with the same second component \( G_2^d \) are transformed into points having the same first component \( G_1 \).

In Figure 2, consider the limit point \( P \) on the left of \( G^d \), which belongs to zone 4 and zone 2. As it has already been shown that \( P \) maintains its abscissa, it turns out that \( G^d \) is transformed into a point having the same first coordinate as \( P \). By symmetric reasoning, it is transformed into a point having the same second coordinate as \( Q \). Figure 2 shows how point \( G \) is obtained from \( G^d \) by constructing the rectangle \( G^dPQG \).

Two phenomena appear in zone 4: first, none of the commodities is completely invested (point \( G \) is inside the STU zone); second, the simultaneous rise of the two desired rates reduces the actual rates, and it is even possible that the rate of accumulation in one of the sectors is negative. To discard such a phenomenon, let us modify the short-side rule and admit that the capitalists obey the following ‘moral’ rule: each entrepreneur commits himself to put on the market at least the quantity supplied during the previous period. The rational selfish basis of this norm is that each capitalist is aware that his long-term interest is not to suffocate the other sector.

II.3. TWO DYNAMICS

This section studies two dynamics driven by the capitalists’ desire for accumulation. In the first case, the desired rates at each period are exogenous and the evolution of the system is defined independently of prices and rates of profit; in the second, the desired rates depend on profits, therefore the evolutions in real and in value terms are interlinked.
II.3.1. Exogenous rates of accumulation

Let the desired rates of accumulation be exogenously given in each sector, feasible at the initial period and constant. If these rates are equal, the system follows a regular growth path. Otherwise, as long as these rates remain feasible, the repetition of the same decisions modifies the structure of the productive system. Assume for instance $g_1^d < g_2^d$, with $g_i = g_i^d$ for each sector $i$. The surplus rate of wheat diminishes while that of iron rises. At a certain date $t$, the feasibility constraint is no longer met because of an excess demand on the wheat market. The short-side rule states that the metallurgist adapts himself by diminishing his accumulation. What is the new rate of accumulation? The level $g_2$ is too high; if $g_2' = g_1$ was adopted, the data of the former period, where the rates were feasible and the wheat still in excess, would be repeated up to the factor $1 + g_1$. Therefore $g_2'$ is between $g_1$ and $g_2$. At the next period, the new reproduction constraint on wheat leads to a new drop of the rate of accumulation in metallurgy. The new rate $g_2^*$ is equal to $g_1$, with a homothetic growth with regard to the previous state: this rate is feasible and, since wheat is wholly accumulated, a higher rate is excluded. At the end of this first crisis, the economy follows a regular growth path at a rate equal to the minimum of the initial rates. In conclusion, if the desired rates are exogenous and constant, this path corresponds to a state of permanent crisis; if the desired rate is equal to the last effective rate ($g_i^d(t) = g_i(t-1)$), the crisis lasts for two periods exactly.

Consider the system in terms of value, within the framework of the two models developed in the first part of the paper. For the same physical evolution, prices and rates of profit differ according to the model.

In the first model, the rate of profit of a sector in each period is equal to the rate of surplus of the corresponding good in the following period. When the rates of accumulation are initially unequal ($g_1 < g_2$), the rate of profit in agriculture falls and that of metallurgy rises progressively until the feasibility constraint on the wheat market is met. The relative price follows the movement of the rates of profit. Once the rates of accumulation are equalized, the rates of profit are set at different levels and remain constant on the regular growth path. Therefore, the equilibrium of accumulation coexists with the disequilibrium of profitability.

In the second model, the sectoral rates of profit are proportional to the actual rates of accumulation. The rates of profit and the prices are initially constant, change during two periods, and then are once again constant. On the regular growth path, the rates of profit are equal and the economy is in full equilibrium (uniform rates of growth and of profit).
II.3.2. Endogenous dynamics

A simple hypothesis is that the desired rates of accumulation represent a given and uniform fraction of last profits.

In the first model, if the profits are totally invested, the desired rates of accumulation are not feasible. The short-side rule reduces these rates to the lower rate of surplus, and the economy is immediately in a state of regular growth with constant rates of profit and prices. On the contrary, if the invested part of profits is smaller than one, the rates of surplus, and hence the successive rates of profit and the desired rates of accumulation, tend to diverge. The dynamic adjustment is obtained by reduction at the lower rate, as in the case of exogenous rates of accumulation, but now the rate itself decreases (it remains nonnegative if the capitalists respect the moral requirement imposed by the modified short-side rule). Consequently, the economy tends towards a stationary state that is reached when the surplus of the sector with the lower rate of profit vanishes.

In the second model, during the first ‘decade’, the desired rates are feasible. As the effective rates of profit are proportional to the rates of accumulation, the path is characterized by constant rates of profit and accumulation. (The dynamics are then similar to those described in the previous section.) The prices and the rates of profit are the unique solution of the equations (5) and (14) adapted to our bisector economy. However, if the two rates of accumulation differ initially \(0 < g_1 < g_2\), for example), the activity levels of the two sectors diverge. This decade of calm comes to its end when, at a certain date, the physical system meets the reproduction constraint on wheat. According to the short-side rule, the constraint weighs on the metallurgist which must reduce his accumulation. It is after this date that the dynamics differ from those considered in the previous sub-section and depend precisely upon the hypothesis retained for the agents’ behaviour. For the farmer, who is not constrained on any market, the unforeseen rise of his rate of profit and of the relative price of wheat (resulting from the relative increase of his rate of accumulation) constitute tangible effects of the change of regime. This change leads him to increase his rate of accumulation, giving rise to an increase in wheat supply. A virtuous circle of increasing profit and accumulation rates in agriculture and their reduction in metallurgy begins. The exact duration of the transition depends on the farmers’ reaction to the windfall profits. The transition is over when the proportions between sectors have changed sufficiently, allowing both capitalists to fulfil their desired rates of accumulation. A new phase of calm begins in which the profit rates, the accumulation rates and the prices remain constant, even if they have changed with regard to
the first phase. This second period will end in a reproduction crisis on the iron market. Thus
the economic system grows, with alternate crises on the markets.

CONCLUSION
Extending Sraffa’s model has allowed us to formalize the notion of reproduction prices and to
build two models in which equilibrium and disequilibrium are dealt with in a unique
framework. The driving variables are the capitalists’ decisions of accumulation, the result of
which is ultimately evaluated by the rates of profit. No hypothesis is made on the use of the
non-accumulated portion of production which acts as a buffer. The models determine the
rates of profit, the relative prices and, depending on the model, other endogenous variables.
The sector rates of profit are positively correlated to the relative rates of accumulation. They
admit a physical interpretation as they are the accumulation rates in the ‘core of
accumulation’ associated with the system. In both models the full equilibrium solution is that
of Sraffa’s model.

Starting from a static model with given quantities, we have sketched two dynamics led by
the desired rates of accumulation, which can depend on profits. The dynamics depend on the
specification of the static model and on the hypotheses that link one period to the other.
According to the case, the economy can evolve towards a regular growth with positive but
weak rate, a cyclical growth with periodic crises or a stationary state.

Some of our assumptions are restrictive. We have not taken into account the technical
change that would allow softening crises. But it is noteworthy that crises occur even when a
strong form of Say’s law is admitted and in the absence of a distinction between capital goods
and consumption goods.

By its prospects and its concepts, this work fits with a Classical approach and is an
attempt to contribute to a dynamic theory of disequilibrium.

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