Fragility of Competitive Equilibrium  
with Risk of Default∗

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ABSTRACT
We study competitive equilibrium in sequential economies under limited commitment. Default induces permanent exclusion from financial markets and endogenously determined solvency constraints prevent debt repudiation. Our analysis shows that such an enforcement mechanism is essentially fragile, leading to equilibrium multiplicity. We accomplish this by establishing Welfare Theorems under a weaker notion of constrained efficiency, inspired by Malinvaud, corresponding to the absence of welfare improving feasible redistributions over finite (though indefinite) horizons. A Negishi’s Method permits to show that, for any arbitrary value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value. Thus, competitive equilibria might differ dramatically in terms of volumes of trade, asset price volatility, individuals’ ability to insure against idiosyncratic risk and consumption inequality.

KEYWORDS. Limited commitment; solvency constraints; Malinvaud efficiency; Welfare Theorems; Negishi’s Method; indeterminacy; market collapse.

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1. Introduction

An important fraction of debt in contemporary economies is essentially unsecured, not backed by any form of collateral. For example, in the US, this is approximately one-third of households debt (over 7% of GDP), mostly due to revolving credit card loans. A second well-known example is sovereign debt, over 80% of GDP in most industrialized countries. Unsecured debt is, in general, enforced through some form of exclusion from financial markets, either based on a reputational mechanism, as for sovereign debt, or on specific regulatory prescriptions, as for most private loans.

In this paper, we study debt enforcement within a large class of competitive economies under sequentially complete markets. Debtors might not deliver on their promises. Upon default, all assets are seized and the debtor is punished by a permanent exclusion from market participation, a complete loss of reputation (Eaton and Gersovitz [15], Kehoe and Levine [17, 18], Kocherlakota [20] and Alvarez and Jermann [4]). At equilibrium, default is prevented by endogenously determined debt limits, quantitative bounds specific to individuals and contingencies, which enforce the maximum expansion of risk-sharing subject to individual rationality of debt repayments (Alvarez and Jermann [4]). The possibility of default limits the diversification of idiosyncratic risks at equilibrium, thus enhancing the predictive power of the theory compared with Arrow-Debreu economies (Alvarez and Jermann [5], Krueger and Perri [22], Krueger, Lustig and Perri [23], Ábrahám and Cárceles-Poveda [1]).

The autarchic punishment upon default bears a relevant advantage in terms of analytical tractability. As matter of fact, beyond material feasibility, risk-sharing is restricted by exogenous reservation utilities, as the autarchic welfare is to be guaranteed to every trader at all contingencies. This delivers a natural form of constrained efficiency, corresponding to the largest feasible risk-sharing subject to participation. Alvarez and Jermann [4] establish a restricted First Welfare Theorem and exploit a Second Welfare Theorem to decentralize constrained efficient allocations. Thus, as in classical macroeconomic analysis, a sort of Negishi’s Method yields equilibrium restrictions on asset prices and consumptions. Yet, this technique of analysis necessarily rules out constrained inefficient equilibria. In particular, constrained efficiency fails when the present value of the aggregate endowment is not finite at equilibrium, or equilibrium does not exhibit high implied interest rates, interpreted as a violation of social transversality.

We expand Alvarez and Jermann’s [4] approach by establishing the two Welfare Theorems for a weak form of constrained efficiency. Malinvaud, or short-run, constrained optimality corresponds to the absence of a welfare-improving feasible redistribution of risk, subject to participation, over any finite horizon (as in Malinvaud [26, 27], Balasko and Shell [8] and Aliprantis, Brown and Burkinshaw [3]). Moreover, we show that there is a vast multiplicity of Malinvaud constrained efficient allocations and, hence, by Welfare Theorems, of competitive equilibria, with social welfare varying from constrained efficiency to autarchy. Finally, we prove that the multiplicity of Malinvaud constrained optima induces indeterminacy: an economy might admit an infinite set of equilibria for a given distribution of initial claims (that is, given contractual obligations inherited from the unrepresented
The validity of Welfare Theorems for Malinvaud constrained optimality clarifies that competitive markets allocate risk efficiently over finite horizons, though they do not impose any discipline in terms of long-run efficiency. This failure of long-run efficiency dramatically affects volumes of trade, price volatility, individuals' ability to insure against idiosyncratic risk and consumption inequality. It is worth noticing that this large variety of equilibrium behaviors is not caused by an exogenous variation of credit constraints, or by a decreasing severity of the punishment for default, as market exclusion remains the implication of debt repudiation and debt limits adjust endogenously.

Multiplicity and indeterminacy of equilibria are proved by developing an extended Negishi's Method that resembles the canonical approach in macroeconomics (at least since Bewley [12] and Kehoe, Levine and Romer [19]). Indeed, Malinvaud constrained efficient allocations can be obtained by the maximization of a utilitarian social welfare function over a restricted domain. This feasible set contains all allocations, satisfying material and participation constraints, attainable by means of finitely many variations of a reference allocation. The restricted social planner program behaves as the unrestricted social planning program, providing a sort of local contract curve, conditional on a given reference allocation. By varying the reference allocation, we show that, globally, there is a vast multiplicity of Malinvaud constrained optima and, hence, by Welfare Theorems, of competitive equilibria. In particular, given any arbitrary value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value (i.e., there is a continuum of equilibria with welfare declining from constrained efficiency to autarchy, provided that the latter is constrained inefficient). Finally, we prove that each local contract curve contains an equilibrium with null initial claims. Hence, globally, there exists an infinite set of equilibria for that given distribution of initial claims. Yet, not all distributions of initial claims exhibit indeterminacy.

Why is equilibrium multiplicity so pervasive? An intuitive explanation is based on the idea that the debt enforcement mechanism is fragile. Indeed, debt is sustained by reputation and the value of reputation might dissipate over time and contingencies. This dissipation process is governed by a sort of dynamic complementarity (Azariadis, Antinolfi and Bullard [6]): the anticipation of tighter debt limits in the future reduces the current value of market participation and increases the incentive to default; debt limits immediately become tighter, responding to lower participation incentives, and interest rate falls to balance the reduced volume of liabilities. However, this self-fulfilling process ultimately relies on a failure of social transversality. Consequently, any institutional mechanism enforcing social transversality eliminates equilibrium multiplicity. This happens when the intertemporal aggregate endowment is finitely valued (or high implied interest rates prevail at equilibrium), as it would be when a sufficiently productive real asset of infinite-maturity is traded or when outstanding public debt is backed by a non-vanishing stream of real primary public surpluses.

Is indeterminacy an artifact of the particular punishment for default? Our techniques exploit a relevant feature of full exclusion, namely, the fact that reservation
utilities (the private values of default) are exogenous. The approach would straightforwardly extend to other exogenously specified reservation utilities. For instance, when default is punished by a partial confiscation of the endowment, beyond full exclusion, private debts are backed by a share of private endowments and Malinvaud constrained efficiency coincides with (unrestricted) constrained efficiency, thus ensuring determinacy of competitive equilibrium (see appendix B). Partial exclusion requires an alternative method of analysis, as it entails an endogenous determination of reservation utilities at equilibrium and constrained efficiency is not unambiguously defined because of the pecuniary externality (that is, redistributions of risk modify reservation utilities through changes in prices). Bulow and Rogoff [13] and Hellwig and Lorenzoni [16] provide a relevant instance of partial exclusion: debt repudiation inhibits future borrowing, though lending remains unrestricted. Other examples include temporary exclusion for a limited number of periods or permanent exclusion with some probability. Although the dual approach presented in this paper might not be practicable, the dynamic complementarity of debt constraints remains and might induce some form of failure of social transversality, leading to equilibrium multiplicity and indeterminacy.

The paper is organized as follows. We begin with developing an extended example in section 2. By abstraction, this serves to inspire our general analysis. In sections 3 and 4, we lay out the fundamentals of a general multi-agent economy with uncertainty and we define a notion of competitive equilibrium with sequential trades and not-too-tight debt constraints. Some technical aspects in section 4 might be skipped at a first reading without affecting the understanding of remaining parts of the paper. In section 5, we present the Negishi’s Method, establishing Welfare Theorems under Malinvaud constrained efficiency. In section 6, we prove equilibrium indeterminacy for an null distribution of initial claims. All proofs are collected in the appendix.

2. An example economy

For heuristic purposes, we present the basic elements of our analytical method in the canonical example of a Bewley economy, also studied by Azariadis [7], Antinolfi, Azariadis, and Bullard [6] and Bejan and Bidian [10]. Two traders are entitled to alternating endowments over an infinite horizon. As the aggregate endowment is constant over time, full efficiency requires constant consumption for each trader. However, limited commitment prohibits this complete consumption smoothing, because traders would alternately profit from defaulting and reverting to permanent autarchy. Consumption volatility is only incompletely eliminated at a competitive equilibrium.

The economy extends over an infinite horizon under no uncertainty,

$$\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}.$$  

There are only two individuals, called even and odd, $\mathcal{J} = \{e, o\}$. Preferences of individuals are represented by

$$U^i_t (x^i) = \sum_{s \in \mathcal{T}} \beta^s u (x^i_{t+s}),$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the common per-period utility, satisfying canonical properties, and $0 < \beta < 1$ is the common discount factor. Endowments $(e^e, e^o)$ are given by
the two sequences
\[ e^c = (e_h, e_l, e_h, e_l, e_h, e_l, \ldots), \]
\[ e^o = (e_l, e_h, e_l, e_h, e_l, e_h, \ldots), \]
where \( 0 < e_l < e_h \). Hence, aggregate endowment is constant and individual endowments are perfectly negatively correlated. Finally, in order to ensure that trade occurs at a constrained optimum and that an unconstrained optimum is not achieved, we postulate that
\[ \beta u'(e_l) > u'(e_h) \]
and
\[ u(e_h) + \beta u(e_l) > u \left( \frac{e_h + e_l}{2} \right) + \beta u \left( \frac{e_h + e_l}{2} \right). \]

For each trader \( i \) in \( J \), the sequential budget constraint imposes
\[ p_{t+1} v_{t+1}^i + p_t (x_{t+1}^i - e_{t+1}^i) \leq p_t v_t^i, \]
where \((p_t)_{t \in T}\) denotes the sequence of present-value prices, \((e_{t+1}^i)_{t \in T}\) is the endowment and \((v_t^i)_{t \in T}\) represents financial wealth (a claim, when positive, and a liability, when negative). The initial wealth, \(v_0^i\), is predetermined, as it is inherited from the unrepresented past. The holding of claims \((v_t^i)_{t \in T}\) is restricted by a solvency constraint,
\[ -f_t^i \leq v_t^i, \]
where the sequence \((f_t^i)_{t \in T}\) represents positive quantitative limits to private liabilities.

At a competitive equilibrium, markets clear and debt limits are endogenously determined so as to ensure market participation. Market clearing for the real market requires a sequence of transfers \((\xi_t)_{t \in T}\) from the high-income to the low-income individual in each period, restricted by
\[ -e_l \leq \xi_t \leq e_h. \]
Market clearing for the financial market corresponds to a sequence \((v_t)_{t \in T}\) representing the amount of wealth held by the low-income individual in every period (compensated by the liability held by the high-income individual). Debt limits are determined by the additional restriction
\[ J_t^i \left( -f_t^i, f \right) = U_t^i \left( e^o \right), \]
where \( J_t^i \left( -f_t^i, f \right) \) denotes the value of market participation, subject to debt limits, beginning with the maximum amount of debt. This participation constraint is justified by limited commitment: a trader cannot profit from repudiating the debt and reverting to permanent autarchy. Importantly, current debt limits cannot be determined independently of future debt limits.

Optimality of consumption plans, subject to budget and debt constraints, requires the following first-order conditions:
\[ \frac{\beta u'(x_{t+1}^i)}{u'(x_t^i)} - \frac{p_{t+1}}{p_t} \leq 0, \]
\[ (v_{t+1}^i + f_{t+1}^i) \left( \frac{\beta u'(x_{t+1}^i)}{u'(x_t^i)} - \frac{p_{t+1}}{p_t} \right) = 0. \]
Basically, a substitution of current consumption for future consumption at market interest rate, by means of additional saving or a reduction of debt, is not profitable; also, the reverse arbitrage, by means of a contraction of savings or additional debt, cannot increase utility when more debt is permitted by the solvency constraint. Obviously, an increase in current consumption, balanced by a reduction of future consumption, might be beneficial when the solvency constraint is binding. Furthermore, the transversality condition imposes
\[
\lim_{t \to \infty} \beta^t u' \left( x_t \right) \left( u'_i f^t_i + f^t_i \right) = 0.
\]
That is, a permanent contraction of asset accumulation, for additional current consumption, is not profitable. First-order conditions, along with the transversality, are necessary and sufficient for optimality. Finally, by the endogenous determination of debt limits, a trader needs be at the autarchic utility whenever the solvency constraint is binding, that is,
\[
\begin{align*}
\left( U_{t+1}^i \left( x^{i+1}_t \right) - U_t^i \left( e^i_t \right) \right) \left( \frac{\beta u' \left( x_{t+1}^i \right)}{u' \left( x_t^i \right)} - \frac{p_{t+1}}{p_t} \right) &= 0.
\end{align*}
\]
This equilibrium characterization is provided by Alvarez and Jermann [4].

We now consider allocations where the high-endowment trader is constrained in every period. Positive transfers \( (\xi_t)_{t \in T} \) from the high-endowment to the low-endowment individual are restricted by
\[
u \left( e_h - \xi_t \right) + \beta u \left( e_l + \xi_{t+1} \right) = u \left( e_h \right) + \beta u \left( e_l \right).
\]
Under the stated assumptions, this implicit difference equation admits a steady state \( \xi^* > 0 \) and a continuum of other solutions, monotonically decreasing to autarchy, for every initial condition \( \xi_0 \) in \( (0, \xi^*) \). At each of such solutions, identified with an initial value \( \xi_0 \) in \( (0, \xi^*) \), corresponds a sequence of implicit prices satisfying
\[
p_{t+1} = \left[ \frac{\beta u' \left( e_l + \xi_{t+1} \right)}{u' \left( e_h - \xi_t \right)} \right] p_t.
\]
Indeed, the market interest rate coincides with the subjective interest rate of the unconstrained individual (the low-income individual).

We now decentralize these allocations as competitive equilibria. Thus, let \( (v_t)_{t \in T} \) denote the sequence of claims held by the unconstrained individual. By market clearing, \(-v_t\) corresponds to the debt of the constrained individual. Budget constraint imposes
\[
p_{t+1} v_{t+1} = p_t \left( \xi_t - v_t \right).
\]
Such a difference equation generates a unique solution, \( (v_t)_{t \in T} \), for any arbitrary initial value \( v_0 \). However, only positive solutions are admitted, for a negative value of \( v_t \) requires a negative debt limit (i.e. forced saving), which we ruled out by assumption. By means of an abstract argument, in the general case, we show that an admissible solution exists.

To summarize, we illustrated the fragility of a debt enforcement mechanism, based on market exclusion, in the context of a simple economy. When endowments are sufficiently dispersed, autarchy is constrained inefficient, and the economy admits a constrained optimal steady state and a continuum of other constrained inefficient equilibria, one for each initial value \( \xi_0 \) in \( (0, \xi^*) \). The non-stationary equilibria
are characterized by a progressive contraction of asset trades and a monotonic convergence of transfers to the autarchic values (i.e., a monotonic increase of individual consumption volatility). It seems that, once uncertainty is introduce, the value of reputation might dissipate at some contingencies and preserve at other contingencies, with a complete collapse of markets occurring only with positive (and possibly extremely low) probability, in the spirit of the recent rare event doctrine for the understanding of asset prices (for instance, Barro [9]). However, in general, examples under less restrictive hypotheses are extremely hard to handle even for constrained efficient equilibria (for instance, Ljungqvist and Sargent [25, Chapter 20.11]). Thus, in the rest of the paper we generalize our analysis, and confute the fragility conjecture, adopting a more abstract approach to equilibrium characterization.

3. Fundamentals

3.1. Time and uncertainty. Time and uncertainty are represented by an event-tree $S$, a countably infinite set, endowed with partial ordering $\succeq$. For a date-event $\sigma$ in $S$, $t(\sigma)$ in $T = \{0, 1, 2, \ldots, t, \ldots\}$ denotes its date and $\sigma_+ = \{\tau \in S(\sigma) : t(\tau) = t(\sigma) + 1\}$ is the non-empty finite set of all immediate direct successors, where $S(\sigma)$ is the set of all date-events $\tau$ in $S$ (weakly) following date-event $\sigma$ in $S$. The initial date-event is $\phi$ in $S$, with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every $\sigma$ in $S$; the initial date-event in $S(\sigma)$ is $\sigma$ in $S$. This construction is canonical (Debreu [14, Chapter 7]).

3.2. Vector spaces. We essentially adhere to Aliprantis and Border [2, Chapters 5-8] for terminology and notation. The reference vector space is $L = \mathbb{R}^S$, the space of all real-valued maps on $S$, with typical element $v = (v_\sigma)_{\sigma \in S}$. The vector space $L$ is endowed with the canonical order: an element $v$ of $L$ is positive if $v_\sigma \geq 0$ for every $\sigma$ in $S$; it is strictly positive if, for some $\epsilon > 0$, $v_\sigma \geq \epsilon$ for every $\sigma$ in $S$. For a positive element $v$ of $L$, we simply write $v \geq 0$ and, when $v$ in $L$ is also non-null, $v > 0$. An element $v$ of $L$ is bounded if, for some $\epsilon > 0$, $|v_\sigma| \leq \epsilon$ for every $\sigma$ in $S$; it is summable if $\sum_{\sigma \in S} |v_\sigma|$ is finite; it is eventually vanishing if $\{\sigma \in S : |v_\sigma| > 0\}$ is a finite subset of $S$. The vector subspace of $L$, consisting of all eventually vanishing elements $v$ of $L$, is denoted by $C$. Finally, the vector space $L$ is endowed with the product topology.

3.3. Individuals. There is a finite set $J$ of individuals. For every individual $i$ in $J$, the consumption space $X^i$ is the positive cone of the commodity space $L$. A consumption plan $x^i$ in $X^i$ is interior (respectively, bounded) if it is uniformly strictly positive (respectively, bounded). An allocation is a distribution of consumption plans across individuals. The space of allocations is

$$X = \{x \in L^J : x^i \in X^i \text{ for every } i \in J\}.$$  

An allocation $x$ in $X$ is interior (respectively, bounded) if every consumption plan $x^i$ in $X^i$ is interior (respectively, bounded).
3.4. **Endowments.** For every individual \( i \) in \( J \), the *endowment* \( e^i \) in \( X^i \) is interior and bounded. In particular, there exists a sufficiently small \( 1 > \epsilon > 0 \) satisfying, at every date-event \( \sigma \) in \( S \),

\[
\epsilon \leq \bigwedge_{i \in J} e^i_{\sigma} \leq \bigvee_{i \in J} e^i_{\sigma} \leq \frac{1}{\epsilon} \text{card}(J),
\]

where \( \text{card}(J) \) in \( \mathbb{N} \) denotes the cardinality of \( J \). This hypothesis imposes a uniform lower bound on the endowment of individuals and, across individuals, an upper bound on the aggregate endowment.

3.5. **Preferences.** We allow for heterogeneous impatience and subjective beliefs, retaining time-additivity of intertemporal utilities in order to simplify the presentation. For every individual \( i \) in \( J \), the per-period utility function \( u^i : \mathbb{R}_+ \to \mathbb{R} \) is bounded, continuous, continuously differentiable, strictly increasing and strictly concave. (As far as smoothness is concerned, more precisely, the per-period utility function is continuously differentiable on \( \mathbb{R}_++ \).) For every individual \( i \) in \( J \), the utility function \( U^i : X^i \to \mathbb{R} \) is given by

\[
U^i(x^i) = \sum_{\sigma \in S} \pi^i_{\sigma} u^i(x^i_{\sigma}),
\]

where \( \pi^i \) is a strictly positive summable element of \( L \). Also, for any date-event \( \sigma \) in \( S \), at any consumption plan \( x^i \) in \( X^i \),

\[
U^i_{\sigma}(x^i) = \frac{1}{\pi^i_{\sigma}} \sum_{\tau \in S(\sigma)} \pi^i_+ u^i(x^i_{\tau}).
\]

This is the continuation utility beginning from date-event \( \sigma \) in \( S \).

3.6. **Uniform impatience.** We impose a uniform bound on the marginal rate of substitution of perpetual future consumption for current consumption. This hypothesis implies a uniform form of impatience across individuals and date-events (see, for instance, Levine and Zame [24, Assumption 5] or Santos and Woodford [29, Assumption 2]). Basically, there exists a sufficiently small \( 1 > \eta > 0 \) satisfying, for every individual \( i \) in \( J \), at every date-event \( \sigma \) in \( S \),

\[
\pi^i_{\sigma} \geq \eta \sum_{\tau \in S(\sigma)} \pi^i_+.
\]

3.7. **Boundary conditions.** This additional hypothesis ensures interiority. For every individual \( i \) in \( J \), at every date-event \( \sigma \) in \( S \),

\[
\eta u^i(0) + (1 - \eta) u^i(1) < u^i(\epsilon),
\]

where \( 1 > \epsilon > 0 \) is given by the bounds on endowments and \( 1 > \eta > 0 \) by the hypothesis of uniform impatience.

3.8. **Subjective prices.** At an interior consumption plan \( x^i \) in \( X^i \), the subjective price \( p^i \) in \( P^i \) is defined by

\[
(p^i_\sigma)_{\sigma \in S} = (\pi^i_{\sigma} \partial u^i(x^i_{\sigma}))_{\sigma \in S}.
\]

The subjective price \( p^i \) in \( P^i \) is a strictly positive summable element of \( L \).
3.9. Feasible allocations. An allocation $x$ in $X$ is feasible if it exhausts aggregate resources and satisfies participation constraints, that is,

$$\sum_{i \in J} x^i = \sum_{i \in J} e^i$$

and, for every individual $i$ in $J$, at every date-event $\sigma$ in $S$,

$$U^i_\sigma(x^i) \geq U^i_\sigma(e^i).$$

The space of all feasible allocations is denoted by $X(e)$. Notice that feasibility reflects both material constraints and participation constraints.

Under the maintained assumptions on preferences and endowments, every feasible allocation is, as a matter of fact, an interior allocation. The particular form of boundary conditions, which is a joint restriction on preferences and endowments, guarantees interiority of consumptions, subject to participation constraints, avoiding unbounded per-period utilities and, hence, simplifying the presentation.

**Lemma 1** (Interiority). *Every feasible allocation is interior.*

4. Equilibrium

Trade occurs sequentially. In every period, a full spectrum of elementary Arrow securities is available for trade, each of which promising a unitary payoff, contingent on the occurrence of a distinct event in the following period. The asset market is, thus, sequentially complete. It simplifies to represent implicit prices of contingent commodities in terms of present values. They are denoted by $p$ in $P$, the space of all strictly positive elements of $L$. At every date-event $\sigma$ in $S$, a portfolio, with deliveries $v$ in $L$ at the following date-events, has a market value, in terms of current consumption, given by

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma^+} p_\tau v_\tau.$$

It should be remarked that, at a price $p$ in $P$, the present value of an arbitrary bounded consumption plan is not necessarily finite.

An individual $i$ in $J$ participates into financial markets. The holding of securities is represented by a financial plan $v^i$ in $V^i$, the space of all unrestricted elements of $L$. Positive values correspond to claims, whereas negative values are liabilities. This participation occurs subject to a sequential budget constraint imposing, at every date-event $\sigma$ in $S$,

$$\sum_{\tau \in \sigma^+} p_\tau v^i_\tau + p_\sigma (x^i_\sigma - e^i_\sigma) \leq p_\sigma v^i_\sigma.$$

Accumulated wealth serves to finance current consumption, in excess to current endowment, and current net asset positions (claims or liabilities). Participation into financial markets is further restricted by quantitative limits to private liabilities. These debt limits are given by $f^i$ in $F^i$, the set of all positive and bounded elements of $L$. The financial plan $v^i$ in $V^i$ is subject, at every date-event $\sigma$ in $S$, to a debt (or solvency) constraint of the form

$$-f^i_\sigma \leq v^i_\sigma.$$

From the perspective of the individual, debt limits are given exogenously.

As in Eaton and Gersovitz [15], Kehoe and Levine [17], Kocherlakota [20] and Alvarez and Jermann [4], commitment is limited. Individuals might not honor
their debt obligations, even though the material availability of future endowments would suffice for a complete repayment. When debt is repudiated, assets are seized and the individual is excluded from future participation into financial markets, though maintaining claims into future uncertain endowment. Thus, unhonored debt induces a permanent reverse to autarchy. At equilibrium, debt limits serve to guarantee that, on the one side, debt repudiation is not profitable for individuals and, on the other side, the maximum sustainable development of financial markets is enforced. This is the notion of equilibrium with not-too-tight debt constraints provided by Alvarez and Jermann [4].

Formally, an allocation $x$ in $X$ is an equilibrium allocation if there exist a price $p$ in $P$, debt limits $f$ in $F$ and financial plans $v$ in $V$ satisfying the following properties:

(a) For every individual $i$ in $J$, the plan $(x^i, v^i)$ in $X^i \times V^i$ is optimal subject to budget and debt constraints, given initial claims, that is, it maximizes intertemporal utility subject, at every date-event $\sigma$ in $S$, to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma (\bar{x}_\sigma^i - \bar{e}_\sigma^i) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$- (\bar{v}_\tau^i + f_\tau^i)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth $v^i_\phi$ in $\mathbb{R}$.

(b) Commodity and financial markets clear, that is,

$$\sum_{i \in J} x^i = \sum_{i \in J} e^i \text{ and } \sum_{i \in J} v^i = 0.$$

(c) For every individual $i$ in $J$, debt limits are not-too-tight, that is, at every date-event $\bar{\sigma}$ in $S$,

$$J_{\bar{\sigma}}^i (-f_{\bar{\sigma}}^i; f^i) = U_{\bar{\sigma}}^i (e^i),$$

where

$$J_{\bar{\sigma}}^i (w_{\bar{\sigma}}^i; f^i) = \sup_{\bar{x}^i} U_{\bar{\sigma}}^i (\bar{x}^i)$$

subject, at every date-event $\sigma$ in $S (\bar{\sigma})$, to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma (\bar{x}_\sigma^i - \bar{e}_\sigma^i) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$- (\bar{v}_\tau^i + f_\tau^i)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth $w_{\bar{\sigma}}^i$ in $\mathbb{R}$. (By convention, the supremum over an empty set is negative infinity.)

Notice that, at equilibrium, for every individual $i$ in $J$, at every date-event $\sigma$ in $S$,

$$U_{\sigma}^i (x^i) = J_{\sigma}^i (v^i_{\sigma}; f^i) \geq J_{\sigma}^i (-f_{\sigma}^i; f^i) = U_{\sigma}^i (e^i).$$

Hence, an equilibrium allocation $x$ in $X$ is, as a matter of fact, an element of $X (e)$, the space of feasible allocations.

We adopt a restrictive notion of equilibrium: first, we require debt limits to be positive; second, we exclude speculative bubbles. Possibly negative debt limits, which are allowed by Alvarez and Jermann [4], would constrain individuals to hold
positive wealth at some contingencies; by defaulting at those contingencies, a trader would in fact refuse a net payment; the enforcement mechanism would be unnatural: A trader would certainly profit from a voluntary exclusion from markets after receiving (and, hence, accepting) the net payment. Furthermore, negative, and possibly unbounded, debt limits would sustain speculative bubbles at equilibrium, a circumstance that has been shown by Kocherlakota [21], based on some properties of homogeneity of the budget set. Finally, notice that debt limits are consistent at equilibrium (according to the terminology borne out by Levine and Zame [24]). Debt limits are consistent when the maximum amount of debt can be honored by means of current endowment and by issuing future contingent debt up to the limit, that is, for every individual \(i\) in \(J\), at every date-event \(\sigma\) in \(S\),

\[
p_\sigma f^i_\sigma \leq p_\sigma e^i_\sigma + \sum_{\tau \in \sigma^+} p_\tau f^i_\tau.
\]

5. Extended Negishi’s Method

5.1. Malinvaud efficiency. Malinvaud efficiency is inherited from studies on capital theory (e.g., Malinvaud [26, 27]) and overlapping generations economies (e.g., Balasko and Shell [8]). The canonical notion of Pareto efficiency requires the absence of a welfare improvement, subject to material and participation constraints. Thus, an allocation \(x\) in \(X(e)\) is Pareto (constrained) efficient if it is not Pareto dominated by an alternative allocation \(z\) in \(X(e)\). The notion of Malinvaud efficiency, instead, imposes weaker restrictions, as it simply requires the absence of a welfare improvement, subject to material and participation constraints, over any arbitrary finite horizon. Consistently, an allocation \(x\) in \(X(e)\) is Malinvaud (constrained) efficient if it is not Pareto dominated by an alternative allocation \(z\) in \(C(e,x)\), where

\[
C(e,x) = \left\{ z \in X(e) : \sum_{i \in J} |z^i - x^i| \in C \right\}
\]

is the set of all allocations \(z\) in \(X(e)\) that modify allocation \(x\) in \(X(e)\) only over a finite horizon. (Remember that \(C\) is the set of all eventually vanishing elements of \(L\).) Clearly, any Pareto optimum is a Malinvaud optimum. However, Malinvaud optimality is a largely weaker requirement: for instance, any autarchic allocation is a Malinvaud optimum. Indeed, as only variations of consumption over finite horizons are allowed, no redistribution satisfies participation constraints at terminal nodes of the finite horizon, because a donor cannot be compensated by the promise of higher continuation utilities. Hence, by induction, no redistribution is the only feasible allocation.

Malinvaud efficiency admits a characterization in terms of supporting price. This is an elaboration on the common duality argument, developed in the literature on capital theory and, more recently, for economies of overlapping generations by Aliprantis, Brown and Burkinshaw [3]. The (algebraic) dual of the vector subspace \(C\) of \(L\) can be identified with \(L\) itself, under the duality operation given, for every \((v,f)\) in \(C \times L\), by

\[
f(v) = f \cdot v = \sum_{\sigma \in S} f_\sigma v_\sigma.
\]
To simplify notation, let
\[ C^* (e, x) = \left\{ z \in X^* (e) : \sum_{i \in J} \left| z^i - x^i \right| \in C \right\}, \]
where \( X^* (e) \) is the set of all allocations \( z \) in \( X \) such that, for every individual \( i \) in \( J \), at every date-event \( \sigma \) in \( S \),
\[ U^i_\sigma (z^i) \geq U^i_\sigma (e^i). \]

Lemma 2 (First-order conditions). An allocation \( x \) in \( X (e) \) is Malinvaud efficient if and only if there exists a price \( p \) in \( P \) satisfying, at every allocation \( z \) in \( C^* (e, x) \), for every individual \( i \) in \( J \),
\[ U^i (z^i) > U^i (x^i) \text{ only if } p \cdot (z^i - x^i) > 0. \]

Equivalently, an allocation \( x \) in \( X (e) \) is Malinvaud efficient if and only if there exists a price \( p \) in \( P \) satisfying, for every individual \( i \) in \( J \), at every date-event \( \sigma \)
\begin{align*}
\text{(c-1)} & \quad \left( \frac{p_\tau}{p_\sigma} \right)_{\tau \in \sigma^+} \geq \left( \frac{p^i_\tau}{p^i_\sigma} \right)_{\tau \in \sigma^+} \\
\text{and} & \\
\text{(c-2)} & \quad \sum_{\tau \in \sigma^+} \left( \frac{p_\tau}{p_\sigma} \right) (U^i_\tau (x^i) - U^i_\tau (e^i)) = \sum_{\tau \in \sigma^+} \left( \frac{p^i_\tau}{p^i_\sigma} \right) (U^i_\tau (x^i) - U^i_\tau (e^i)),
\end{align*}
where \( p^i \) in \( P^i \) is the subjective price at interior consumption plan \( x^i \) in \( X^i \).

Restriction (s) coincides with an admittedly abstract characterization of Malinvaud optima in terms of supporting positive linear functionals, whereas conditions (c-1)-(c-2) uncover an equivalent formulation in terms of more treatable first-order conditions. For the sake of simplicity, though this is unprecise, the above characterization might be illustrated by referring to a canonical social planner problem. Restrictions (c-1)-(c-2) correspond, in this analogy, to the Euler equations induced by the maximization of (weighted) social welfare subject to material constraints and to participation constraints. They basically rule out the circumstance of a constrained individual exhibiting a marginal rate of substitution strictly above the marginal rate of substitution of an unconstrained individual. This, indeed, would expose to an arbitrage opportunity, as a substitution of future consumption for current consumption of the unconstrained individual, balanced by the opposite substitution for the constrained individual, would not violate participation constraint, as utility of the unconstrained individual is strictly above the autarchic utility, and would produce a welfare improvement.

The remarkable implication of this full characterization is that a Malinvaud optimum does not impose any restriction in terms of social transversality or, alternatively, does not rule out any arbitrage opportunity at infinitum. A substitution of current consumption for perpetual future consumption might still generate a welfare improvement, subject to feasibility.
5.2. Social planning. A canonical method in macroeconomic theory permits the study of Pareto efficient allocations by means of a social planning programme. Given welfare weights $\theta$ in $\Theta$, social welfare is measured by the weighted sum of utilities,

$$W_{\theta}(x) = \sum_{i \in J} \theta^i u^i(x^i),$$

where

$$\Theta = \left\{ \theta \in \mathbb{R}_+^J : \sum_{i \in J} \theta^i = 1 \right\}.$$

It is well-established that an allocation $x$ in $X(e)$ is Pareto efficient if and only if there exist welfare weights $\theta$ in $\Theta$ satisfying

$$W_{\theta}(x) = \max_{z \in X(e)} W_{\theta}(z).$$

Various properties of efficient allocations are directly obtained by exploiting this well-behaved convex programme.

To provide an analogous treatment of Malinvaud efficiency, we introduce a restricted feasible set for social planning. Given an allocation $x$ in $X(e)$, define

$$F(e,x) = \text{closure} \left\{ z \in X(e) : \sum_{i \in J} |z^i - x^i| \in C \right\},$$

where the closure is taken in the product topology. Intuitively, this restricted feasible set consists of all allocations $z$ in $X(e)$ that can be approached by means of finitely many variations of a reference allocation $x$ in $X(e)$. Importantly, this restricted feasible set is convex and compact.

**Lemma 3** (Social planning). An allocation $x$ in $X(e)$ is Malinvaud efficient if and only if, for some allocation $y$ in $X(e)$ such that $x$ lies in $F(e,y)$, there exist welfare weights $\theta$ in $\Theta$ satisfying

$$W_{\theta}(x) = \max_{z \in F(e,y)} W_{\theta}(z).$$

The comparison between the Pareto (unrestricted) planner problem and the Malinvaud (restricted) planner problem reveals strong analogies. By a canonical application of the Maximum Theorem, solutions vary continuously with welfare weights. Hence, the restricted planner problem yields a connected set of Malinvaud efficient allocations for any arbitrarily given reference feasible allocation. This, along with Welfare Theorems, provides a potential tool for the computation of competitive equilibria.

5.3. Multiplicity of Malinvaud optima. We now provide a partial characterization of Malinvaud optima. In particular, we prove that there exists a continuum of such optima with social welfare decreasing from Pareto efficiency to autarchy. (Obviously, when the autarchy is Pareto efficient, this multiplicity disappears.) Malinvaud optima are parameterized by welfare weights $\theta$ in $\Theta$ and an index $\xi$ in $\Xi = [0, 1]$ measuring the failure of Pareto (constrained) optimality. Hence, the set of Malinvaud optima contains a set that is isomorphic to $\Theta \times \Xi$. 
Proposition 1 (Multiplicity). Given welfare weights $\theta$ in $\Theta$, for any arbitrary value $\xi$ in $\Xi = [0, 1]$, there exists a Malinvaud efficient allocation $x$ in $X(e)$ with social welfare satisfying

\[(w-1) \quad W_\theta(x) = W_\theta(e) + \xi \left( \max_{z \in X(e)} W_\theta(z) - W_\theta(e) \right)\]

and

\[(w-2) \quad W_\theta(x) = \max_{z \in F(e,x)} W_\theta(z).\]

This simple characterization emerges by means of artificial truncated planner problems, along with a limit argument. A truncation consists in imposing additional restrictions on the amount of redistributed resources that can be implemented out of some finite horizon. For a given truncation, the severity of these additional restrictions determines the value of the social planner problem: under the most severe restrictions, the redistribution vanishes out of a finite horizon and, hence, the autarchy is the only feasible allocation (indeed, a decrease of consumption in the last period of the truncation cannot be compensated by an increase of consumption in the following periods and, hence, by induction, no redistribution is the only feasible policy); under the least severe restrictions, any feasible allocation can be implemented and, hence, a Pareto optimum obtains. It follows that, for any given truncation, some properly chosen degree of severity of additional constraints would yield a given social welfare in between autarchy and Pareto efficiency. Taking the limit over finite horizons, a limit allocation emerges with a given social welfare value (as this can be assumed to be constant along the sequence). This limit allocation is Malinvaud efficient because, as the finite horizon extends along the sequence of truncations, first-order conditions are satisfied along larger and larger horizons.

5.4. Welfare Theorems. We here show equivalence between equilibrium allocations and Malinvaud efficient allocations. Indeed, any equilibrium allocation is Malinvaud efficient (First Welfare Theorem) and any Malinvaud efficient allocation emerges as an equilibrium allocation for some balanced distribution of initial claims (Second Welfare Theorem). As a matter of fact, we prove that Malinvaud efficiency exhausts all restrictions on equilibrium prices and allocations.

Proposition 2 (First Welfare Theorem). Any equilibrium allocation is a Malinvaud efficient allocation.

The First Welfare Theorem is almost immediate. Indeed, first-order conditions for a Malinvaud optimum coincides with those for an equilibrium under limited commitment (see Alvarez and Jermann [4]). At equilibrium, the marginal rate of substitution of an individual falls below the market rate of substitution only if this individual is constrained in issuing further debt obligations, for otherwise a budget-balanced (marginal) substitution of future consumption for current consumption would yield an increase in welfare.

Proposition 3 (Second Welfare Theorem). Any Malinvaud efficient allocation is an equilibrium allocation.

The proof of the Second Welfare Theorem cannot rely on a traditional separation argument alone. Indeed, separation yields potential equilibrium prices fulfilling first-order conditions (lemma 2). Such prices, however, might not belong to the
dual of the commodity space (restricted by the aggregate endowment) and, thus, might not deliver a well-defined intertemporal accounting. In order to provide their Second Welfare Theorem for Pareto efficient allocations, Alvarez and Jermann [4] assume that prices belong to the dual of the (restricted) commodity space (the hypothesis of high implied interest rates) and recover financial plans at equilibrium as the present value of future contingent net trades. We cannot count on this simple method and need an alternative proof. Furthermore, differently from Alvarez and Jermann [4], as well as from Kocherlakota [21], we impose positivity of debt limits (individuals cannot be restricted to hold positive amounts of wealth along the infinite horizon), which poses additional difficulties.

To recover financial plans, we construct an adjustment process that increases debt, when more debt is budget-feasible, and decreases debt, when outstanding debt is budget-unfeasible. This process admits a fixed point and, at the fixed point, sequential budget constraints are balanced and financial markets clear. To identify a suitable interval, we move from a basic observation. We evaluate welfare gains, with respect to the autarchic utility, in terms of current consumption. Participation guarantees that these welfare gains are positive across date-events. Also, they fulfill sequential budget constraints at subjective prices (marginal utilities). As market rates of substitution differ from individual marginal rates of substitution only when welfare gains vanish, the process of welfare gains also satisfies sequential budget constraints at market prices. This yields an upper bound on the amount of wealth held at equilibrium, as welfare gains are positive (hence, fulfil debt limits) and sustains the given consumption plan subject to sequential budget constraints. Since financial plans need be balanced at equilibrium across individuals, the negative of the sum of welfare gains poses a lower bound to financial plans.

Optimality of consumption plans, subject to budget constraints and debt constraints, is ensured by first-order conditions at a Malinvaud optimum. Hence, it only remains to reconstruct suitable debt limits. Here, we follow Alvarez and Jermann [4]. When an individual is at the autarchic utility, outstanding debt coincides with the maximum amount of debt. When an individual is not at the autarchic utility, we compute the maximum amount of sustainable debt, which depends on the future contingent plan for debt limits. Beginning with sufficiently large debt limits, this process of adjustment generates a decreasing sequence of debt limits and, in the limit, we obtain not-too-tight debt constraints. The identification of suitable upper bounds requires some elaboration.

6. Indeterminacy

Indeterminacy pertains to the existence of an infinite set of competitive equilibria for given endowments and given (balanced) distribution of initial claims. It is preliminarily worth remarking that initial claims, or liabilities, are real and that debt limits are restricted to be positive at equilibrium. Therefore, indeterminacy is not a nominal phenomenon, caused by the unavoidable multiplicity of the price level, as in monetary overlapping generations economies. Furthermore, it is not due to the occurrence of bubbles, uncovered by Kocherlakota [21], which would require that traders be forced to accumulate assets by negative debt limits (i.e., mandatory savings). As a matter of fact, unrestricted debt limits would immediately induce indeterminacy by homogeneity of budget sets, independently of the
established multiplicity of Malinvaud optima: a given equilibrium allocation would be consistent with any arbitrary distribution of initial claims.

The intuition for indeterminacy of competitive equilibrium is given by the fundamental characterization of Malinvaud optima (proposition 1). Indeed, welfare weights account for a merely distributive multiplicity, reflecting the distribution of initial claims. It remains an autonomous degree of multiplicity, interpretable as a measure of the reputation of debtors, varying from values enforcing the development of financial markets (efficiency) to the complete collapse of financial markets (autarchy). Beyond intuition, common techniques of analysis in the literature are not directly applicable and a formal proof is not straightforward.

Consider the space of balanced distributions of initial wealth,

\[ \Omega = \{ \omega \in \mathbb{R}^J : \sum_{i \in J} \omega_i = 0 \} . \]

Also, let \( E_{\omega} (e) \) denote the set of equilibrium allocations \( x \) in \( X (e) \) which are consistent with the distribution of initial claims \( \omega \) in \( \Omega \). By means of the adapted Negishi’s Method, we establish the existence of equilibria with a null initial distribution (neither claims nor liabilities inherited from previous contractual arrangements).

**Lemma 4 (Existence).** Given any arbitrary allocation \( y \) in \( X (e) \),

\[ \Omega (e, y) = \{ \omega \in \Omega : E_{\omega} (e) \cap M (e, y) \text{ is non-empty} \} \]

is a connected set containing the null distribution of initial claims in \( \Omega \), where

\[ M (e, y) = \left\{ x \in F (e, y) : W_{\theta} (x) = \max_{z \in F (e, y)} W_{\theta} (z) \text{ for some } \theta \in \Theta \right\} . \]

The argument unfolds as follows. Within the given component of Malinvaud efficient allocations, every allocation is consistent with some distribution of initial claims by the established Second Welfare Theorem. We adjust welfare weights inversely with the initial claims held by traders. This process admits a fixed point and, at the fixed point, the initial claim of a trader is strictly positive only if the corresponding welfare weight vanishes and, thus, the trader is at the reservation utility. However, this cannot occur, as market participation, subject to positive debt limits, always guarantees a utility strictly higher than autarchy when a trader begins with a strictly positive claim.

This persistence of the null distribution, across components of Malinvaud efficient allocations, reveals the existence of an infinite set of competitive equilibria, with social welfare varying from efficiency to autarchy.

**Proposition 4 (Indeterminacy).** Given the null distribution of initial claims, there exists an infinite set of equilibria, i.e., \( E_{\emptyset} (e) \) is not finite.

To ascertain determinacy, or indeterminacy, of equilibrium for given non-null distributions of initial claims appears technically difficult (in particular, when the supporting welfare weights are extreme points and some individuals are at the autarchic utilities). By means of numerical computations, in the context of the classical example economy described in section 2, we show that the Pareto efficient steady state distribution of initial claims is not consistent with inefficient allocations converging to the autarchy: debt limits should be negative in some periods to sustain such allocations as equilibria (see appendix C). Thus, at least for some
specifications of utility and endowments, the stationary Pareto efficient equilibrium is determinate.

7. Conclusion

We have shown multiplicity, and related indeterminacy, of equilibria in economies with limited enforcement and not-too-tight debt limits. In particular, we have developed a method that exploits Welfare Theorems for deriving a full characterization of equilibria. These theorems are established for a weak form of optimality, corresponding to the absence of a feasible Pareto improving redistribution over a finite number of time periods. These weak optima, in turn, are characterized by means of sequences of planning objectives with limited amounts of redistributions in the long-run. The method shows that, at equilibrium, social welfare varies from two extreme outcomes: constrained Pareto optimality and autarchy.

Our contribution bears very important consequences on the understanding of the type of equilibria that may emerge in economies where contract enforcement is limited and the no default option is implemented by imposing individual specific debt constraints. In particular, these equilibria suffer from a severe form of fragility: a change in expectations at any given equilibrium, where asset trades guarantee an optimal amount of consumption smoothing across states and time periods, might generate a contraction of net trades, in some cases leading to financial collapse.

References

Appendix A. Proofs

Proof of lemma 1. At a feasible allocation, for every individual \( i \) in \( J \), participation constraints impose, at every date-event \( \sigma \) in \( S \),

\[
 u^i \left( x^i_\sigma \right) - u^i \left( \frac{1}{\epsilon} \right) + \frac{1}{\pi^i_\sigma} \sum_{\tau \in S(\sigma)} \pi^i_\tau u^i \left( \frac{1}{\epsilon} \right) \geq U^i_\sigma \left( x^i \right) \geq U^i_\sigma \left( e^i \right) \geq \frac{1}{\pi^i_\sigma} \sum_{\tau \in S(\sigma)} \pi^i_\tau u^i \left( e^i \right) .
\]

Therefore, exploiting uniform impatience and boundary conditions,

\[
 u^i \left( x^i_\sigma \right) \geq u^i \left( \frac{1}{\epsilon} \right) + \frac{1}{\eta} \left( u^i \left( e^i \right) - u^i \left( \frac{1}{\epsilon} \right) \right) > u^i \left( 0 \right),
\]

which produces a uniformly strictly positive lower bound on consumptions. \( \square \)

Proof of lemma 2. Sufficiency of a supporting price \( p \) in \( P \) \( \text{(i.e., condition (s))} \) for Malinvaud efficiency is obvious, as it is proved by the traditional argument for the canonical First Welfare Theorem. Therefore, we show that restrictions (c-1)-(c-2) imply condition (s). Consider any alternative allocation \( z \) in \( C^*(e, x) \) and suppose that, for some individual \( i \) in \( J \),

\[
 0 < U^i \left( z^i \right) - U^i \left( x^i \right) \leq \sum_{\sigma \in S} p^i_\sigma \left( z^i_\sigma - x^i_\sigma \right) .
\]
Define, at every date-event $\sigma$ in $S$,
\[
v_i^\sigma = \frac{1}{p_\sigma} \sum_{\tau \in S(\sigma)} p_\tau^i (z_\tau^i - x_\tau^i).
\]

Notice that $v^i$ is an element of $C$. A simple decomposition yields, at every date-event $\sigma$ in $S$,
\[
(*) \quad \sum_{\tau \in \sigma_+} p_\tau^i v_i^\tau + p_\sigma^i (z_\sigma^i - x_\sigma^i) \geq p_\sigma^i v_i^\sigma.
\]

Furthermore, notice that convexity of preferences and participation constraints imply that, at every date-event $\sigma$ in $S$,
\[
v_i^\sigma < 0 \quad \text{only if} \quad U_i^\sigma (e_i) \leq U_i^\sigma (z_i) < U_i^\sigma (x_i).
\]

Therefore, restrictions (c-1)-(c-2), along with inequality (*), guarantee that, at every date-event $\sigma$ in $S$,
\[
\sum_{\tau \in \sigma_+} p_\tau^i v_i^\tau + p_\sigma^i (z_\sigma^i - x_\sigma^i) \geq p_\sigma^i v_i^\sigma.
\]

Consolidating across date-events, and noticing that $v^i$ is an element of $C$, one obtains
\[
p \cdot (z^i - x^i) = \sum_{\sigma \in S} p_\sigma (z_\sigma^i - x_\sigma^i)
\geq p_\phi v_\phi^i
\geq \left( \frac{p_\phi}{p_\phi'} \right) \left( \sum_{\sigma \in S} p_\sigma^i (z_\sigma^i - x_\sigma^i) \right)
> 0,
\]

thus proving the claim.

Assume now that the allocation $x$ in $X(e)$ is Malinvaud-efficient and define a price $p$ in $P$ by means, at every date-event $\sigma$ in $S$, of
\[
\left( \frac{p_\tau}{p_\sigma} \right)_{\tau \in \sigma_+} = \bigvee_{i \in \mathcal{J}} \left( \frac{p_i^\tau}{p_i^\sigma} \right)_{\tau \in \sigma_+}.
\]
This price $p$ in $P$ obviously satisfies condition (c-1). The necessity of condition (c-2) straightforwardly obtains by means of the argument in Alvarez and Jermann [4, Proposition 3.1]. As conditions (c-1)-(c-2) imply restriction (s), this completes the proof. 

\[\square\]

**Proof of lemma 3.** Preliminarily observe that an allocation $x$ in $X(e)$ is Malinvaud inefficient if and only if it is strictly Pareto dominated by an alternative allocation $z$ in $C(e, x)$. Indeed, any balanced redistribution of initial consumptions does not affect future participation. Suppose that allocation $x$ in $X(e)$ is Malinvaud efficient. Obviously, $x$ lies in $F(e, x)$. A canonical application of the Separation Theorem yields the existence of welfare weights $\theta$ in $\Theta$ such that
\[
W_\theta (x) = \max_{z \in F(e, x)} W_\theta (z).
\]
To prove the reverse claim, it suffices to establish that $C(e, x)$ is contained in $F(e, y)$. Let $z$ be in $C(e, x)$ and consider any sequence $\{x^n\}_{n \in \mathbb{N}}$ in $C(e, y)$ approaching $x$ in $F(e, y)$. For every individual $i$ in $J$, let $F^i$ be the finite subset of all date-events in $S$ at which the reallocation is not terminated, that is,

$$\sigma \in (S/F^i) \text{ if and only if } (z^i_x)_\tau \in S(\sigma) = (x^i_x)_\tau \in S(\sigma).$$

Given $1 > \lambda > 0$, for every sufficiently large $n$ in $\mathbb{N}$, consider the allocation $x^n + \lambda(z - x)$ in $X$. This is a balanced allocation, as consumptions are strictly positive. We show that, for every sufficiently large $n$ in $\mathbb{N}$, it also satisfies participation.

Notice that participation might be violated only on the finite set $F = \bigcup_{i \in J} F^i$.

If it is violated, there exists an individual $i$ in $J$ such that, at some date-event $\sigma$ in $F^i$, for infinitely many $n$ in $\mathbb{N}$,

$$U_i^\sigma (x^n + \lambda(z^i - x^i)) < U_i^\sigma (e^i).$$

In the limit, this implies

$$U_i^\sigma (x^i + \lambda(z^i - x^i)) \leq U_i^\sigma (e^i),$$

which, by convexity of preferences, reveals a contradiction. Hence, for every sufficiently large $n$ in $\mathbb{N}$, allocation $x^n + \lambda(z - x)$ lies in $F(e, y)$ and, hence, $z$ lies in $F(e, y)$, thus proving the claim. □

**Proof of proposition 1.** The proof is decomposed in several separate steps. First, we construct a sequence of truncated planner problems, by adding additional auxiliary constraints on the transfers across individuals; truncated optima exist and, at given welfare weights $\theta$ in $\Theta$, social welfare might be measured by $\xi$ in $\Xi$ by controlling for the severity of additional constraints on transfers. Second, we generate a sequence of truncated optima, maintaining a constant value of social welfare, and we consider the limit allocation of these truncated planner problems. Third, we prove that the limit allocation is in fact a Malinvaud optimum fulfilling conditions (w-1)-(w-2).

**Truncation.** Given any $t$ in $T$, consider a collection of $t$-truncated planner problems:

$$V^\sigma_t (\epsilon) = \max_{x \in X(\epsilon)} \sum_{i \in J} \theta U_i^\sigma (x^i)$$

subject to, at every date-event $\sigma$ in $(S/S^t)$,

$$\sum_{i \in J} |x^i_\sigma - e^i_\sigma| \leq \epsilon,$$

where, for every $t$ in $T$,

$$S^t = \{ \sigma \in S : t(\sigma) \leq t \}.$$

Constraints are given as a continuous correspondence of $\epsilon$ in $\mathbb{R}_+$ with non-empty convex and compact values. (Indeed, notice that the map $x \mapsto |x|$ is convex. In addition, if allocation $x$ in $X(\epsilon)$ satisfies constraints $(\dagger)$ at $\epsilon$ in $\mathbb{R}_{++}$, then allocation

$$x - \left( \frac{\epsilon - \epsilon^*}{\epsilon} \right)^+ (x - e) \in X(\epsilon)$$

satisfies constraints $(\dagger)$ at $\epsilon$ in $\mathbb{R}_{++}$.)
satisfies constraints (†) at $\epsilon^*$ in $\mathbb{R}_+$. Hence, by the Maximum Theorem, the maximum is achieved and the value function is continuous in $\epsilon$ in $\mathbb{R}_+$.

Observe that, when $\epsilon$ in $\mathbb{R}_+$ is sufficiently large, the truncated problem delivers a constrained Pareto efficient allocation, that is, $V_\theta^t(\epsilon) = \max_{x \in X(\epsilon)} W_\theta(x)$; when $\epsilon$ in $\mathbb{R}_+$ vanishes, the truncated problem delivers the autarchy, as this is the only feasible allocation $x$ in $X(\epsilon)$ satisfying additional constraints (†), that is, $V_\theta^t(0) = W_\theta(\epsilon)$.

Hence, by the Intermediate Value Theorem, for some value of $\epsilon$ in $\mathbb{R}_+$,

$$V_\theta^t(\epsilon) = W_\theta(\epsilon) + \xi \left( \max_{x \in X(\epsilon)} W_\theta(x) - W_\theta(\epsilon) \right).$$

Let $x^t$ be an allocation in $X(\epsilon)$ that solves the $t$-truncated planner problem at the value of $\epsilon$ in $\mathbb{R}_+$ fulfilling the above restriction.

\[\square\]

Limit. The sequence of allocation $\{x^t\}_{t \in T}$ in $X(\epsilon)$, at no loss of generality, converges to some allocation $x$ in $X(\epsilon)$ in the product topology. Also, by continuity of preferences, restriction (W-1) is satisfied by the limit allocation $x$ in $X(\epsilon)$.

\[\square\]

Malinvaud optimality in the limit. It remains to verify that condition (W-2) is valid in the limit. Supposing not, there exists an alternative allocation $z$ in $C(\epsilon, x)$ satisfying $W_\theta(z) > W_\theta(x)$. Defining the set $F$ as in the previous part, for every sufficiently large $t$ in $\mathcal{T}$, $F \subset S^t$. Hence, allocation $z$ in $X(\epsilon)$ fulfills the additional restrictions (†) for every sufficiently large $t$ in $\mathcal{T}$. It follows that, for every sufficiently large $t$ in $\mathcal{T}$, $W_\theta(x^t) \geq W_\theta(z)$, yielding a contradiction.

\[\square\]

The sequence of steps proves the proposition.

\[\square\]

Proof of proposition 2. Using lemma 2, Malinvaud efficiency follows from the simple first-order characterization of equilibrium that is provided by Alvarez and Jermann [4, Propositions 4.5-4.6].

\[\square\]

Proof of proposition 3. The proof is rather involved, so that we decompose it in several steps.

Recovering financial plans. To simplify notation, we introduce the positive linear operator $T : L \rightarrow L$ that is defined, at every date-event $\sigma$ in $\mathcal{S}$, by

$$T(v)_\sigma = \frac{1}{p^t_\sigma} \sum_{\tau \in \sigma_+} p^t_\tau v^t_\tau.$$

For an individual $i$ in $\mathcal{J}$, let $g^i$ in $L$ be given, at every date-event $\sigma$ in $\mathcal{S}$, by

$$g^i_\sigma = \left( \frac{\pi^i_\sigma}{p^t_\sigma} \right) \left( U^i_\sigma(x^i) - U^i_\sigma(e^i) \right) \leq \frac{1}{\pi^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p^t_\tau \left( \frac{u^i(1/\epsilon) - u^i(\epsilon)}{\partial u^i(1/\epsilon)} \right).$$

Notice that, by uniform impatience and feasibility, $g^i$ is a bounded and positive element of $L$. Furthermore, observe that, by convexity of preferences, at every date-event $\sigma$ in $\mathcal{S}$,

$$\pi^i_\sigma \left( U^i_\sigma(x^i) - U^i_\sigma(e^i) \right) \geq \pi^i_\sigma \left( u^i(x^i_\sigma) - u^i(e^i_\sigma) \right) + \sum_{\tau \in \sigma_+} \pi^i_\tau \left( U^i_\tau(x^i) - U^i_\tau(e^i) \right) \geq p^i_\sigma \left( x^i_\sigma - e^i_\sigma \right) + \sum_{\tau \in \sigma_+} \pi^i_\tau \left( U^i_\tau(x^i) - U^i_\tau(e^i) \right).$$
This, exploiting first-order conditions at a Malinvaud optimum, yields
\[ T(g^i) + (x^i - e^i) \leq g^i. \]

Finally, define \( g = \sum_{i \in \mathcal{J}} g^i \) and observe that \( g \) is a positive bounded element of \( L \).

Define \( H \) as the set of all \( h \) in \([0, g]^\mathcal{J}\) satisfying
\[ \sum_{i \in \mathcal{J}} h^i = g. \]

The set \( H \) is non-empty, convex and compact (in the product topology). Define a correspondence \( f : H \to H \) by means of
\[ f(h) = \arg \min_{\hat{h} \in H} \sum_{i \in \mathcal{J}} \hat{h}^i \left( T(g^i - \hat{h}^i) + (x^i - e^i) - (g^i - h^i) \right). \]

Basically, if a financial plan lies in the interior of the budget constraint at some date-event, current debt is increased. By construction, given any \( h \) in \( H \), at every date-event \( \sigma \) in \( \mathcal{S} \), there exists an individual \( i \) in \( \mathcal{J} \) such that
\[ T(g^i - h^i) + (x^i - e^i) \leq (g^i - h^i), \]

as
\[ \sum_{i \in \mathcal{J}} (T(g^i - h^i) + (x^i - e^i)) = 0 = \sum_{i \in \mathcal{J}} (g^i - h^i). \]

The correspondence \( f : H \to H \) is clearly closed with non-empty and convex values. Moreover, since \( H \) is compact, \( f \) is also upper hemi-continuous. Therefore, by Kakutani Fixed Point Theorem, it admits a fixed point \( h \) in \( H \). At a fixed point, for every individual \( i \) in \( \mathcal{J} \), at any date-event \( \sigma \) in \( \mathcal{S} \),
\[ T(g^i - h^i) + (x^i - e^i) > (g^i - h^i) \implies h^i = 0. \]

Hence,
\[ g^i \geq T(g^i) + (x^i - e^i) \geq T(g^i - h^i) + (x^i - e^i) > g^i, \]

which is a contradiction. Thus, at a fixed point, for every individual \( i \) in \( \mathcal{J} \),
\[ T(g^i - h^i) + (x^i - e^i) \leq (g^i - h^i). \]

Since, aggregating on \( \mathcal{J} \), \( \sum_{i \in \mathcal{J}} h^i = \sum_{i \in \mathcal{J}} g^i \), the latest suffices to prove exact budget-feasibility, that is, for every individual \( i \) in \( \mathcal{J} \),
\[ T(g^i - h^i) + (x^i - e^i) = (g^i - h^i). \]

To conclude, for every individual \( i \) in \( \mathcal{J} \), the financial plan \( v^i = g^i - h^i \) in \( V^i \) is bounded, balances budget sequentially and satisfies, at every date-event \( \sigma \) in \( \mathcal{S} \),
\[ U^i_{\sigma}(x^i) = U^i_{\sigma}(e^i) \text{ only if } v^i = g^i - h^i \leq g^i \leq 0. \]

Furthermore, across individuals, financial plans \( v \) in \( V \) satisfy market clearing, that is,
\[ \sum_{i \in \mathcal{J}} v^i = 0. \]

We treat such financial plans as given in the remaining parts of this proof. \( \square \)
Individual optimality. For every individual $i$ in $\mathcal{J}$, consider the set of all date-events at which this individual is at the autarchic utility, that is,

$$S^i = \{ \sigma \in \mathcal{S} : U^i_\sigma(x^i) = U^i_\sigma(e^i) \}.$$ 

Also, define the space $F^i(x^i, v^i)$ of all debt limits $f^i$ in $F^i$ satisfying, at every date-event $\sigma$ in $\mathcal{S}$, $v^i_\sigma + f^i_\sigma \geq 0$ and, at every date-event $\sigma$ in $S^i$, $v^i_\sigma + f^i_\sigma = 0$. We here show that consumption plan $x^i$ in $X^i$ is optimal, subject to budget and debt constraints, given initial claims, at all debt limits $f^i$ in $F^i(x^i, v^i)$.

Peg any date-event $\sigma$ in $\mathcal{S}$. Observe that, as budget is balanced,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f^i_\tau - \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (v^i_\tau + f^i_\tau) - (x^i_\sigma - e^i_\sigma) \leq -v^i_\sigma.$$

Furthermore, considering any alternative budget feasible consumption plan $z^i$ in $X^i$ satisfying debt constraints,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f^i_\tau + \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (w^i_\tau + f^i_\tau) + (z^i_\sigma - e^i_\sigma) \leq w^i_\sigma.$$

Using first-order conditions, one obtains

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f^i_\tau - \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (v^i_\tau + f^i_\tau) - (x^i_\sigma - e^i_\sigma) \leq -v^i_\sigma$$

and

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f^i_\tau + \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (w^i_\tau + f^i_\tau) + (z^i_\sigma - e^i_\sigma) \leq w^i_\sigma.$$

Therefore, adding up terms, it follows that

$$(\dagger) \quad \sum_{\tau \in \sigma_+} p_\tau (w^i_\tau - v^i_\tau) + p_\sigma (z^i_\sigma - x^i_\sigma) \leq p_\sigma (w^i_\sigma - v^i_\sigma).$$

For every $t$ in $\mathcal{T}$, let

$$\mathcal{S}_t = \{ \sigma \in \mathcal{S} : t(\sigma) = t \} \quad \text{and} \quad \mathcal{S}^t = \{ \sigma \in \mathcal{S} : t(\sigma) \leq t \}.$$ 

Consolidating inequalities (\dagger) up to period $t$ in $\mathcal{T}$, and using the fact the initial claims are given,

$$\sum_{\sigma \in \mathcal{S}^t} p_\sigma (z^i_\sigma - x^i_\sigma) \leq \sum_{\tau \in \mathcal{S}_{t+1}} p_\tau (v^i_\tau - w^i_\tau) \leq \sum_{\tau \in \mathcal{S}_{t+1}} p_\tau (v^i_\tau + f^i_\tau),$$

where the last inequality follows from debt constraints. By concavity of utility, this suffices to prove optimality, as the right hand-side vanishes in the limit, because $p^i$ in $P^i$ is a summable element of $L$ and $v^i + f^i$ is a bounded element of $L$. \hfill $\Box$

Recovering debt limits. Given debt limits $f^i$ in $F^i$, at every date-event $\tilde{\sigma}$ in $\mathcal{S}$, let $B^i_{\tilde{\sigma}}(w^i_{\tilde{\sigma}}, f^i_{\tilde{\sigma}})$ be the set of all plans $(\tilde{x}^i, \tilde{v}^i)$ in $X^i \times V^i$ satisfying, at every date-event $\sigma$ in $\mathcal{S}(\tilde{\sigma})$, budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \tilde{v}^i_\tau + p_\sigma (\tilde{x}^i_\sigma - e^i_\sigma) \leq p_\sigma \tilde{v}^i_\sigma,$$

and debt constraints,

$$-(\tilde{v}^i_\tau + f^i_\tau)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth $w^i_{\tilde{\sigma}}$ in $\mathbb{R}$. 
Consider the set
\[ D^i = \{(w^i, f^i) : B^i_\sigma(w^i; f^i) \text{ is non-empty at every } \sigma \in S \} \subset V^i \times F^i. \]

This domain is non-empty, closed and convex. Define a value function \( J^i : D^i \to L \) by means, at every date event \( \sigma \in S \), of
\[ J^i_\sigma (w^i; f^i) = \max \{ U^i_\sigma (\bar{x}^i) : (\bar{x}^i, \bar{v}^i) \in B^i_\sigma (w^i; f^i) \}. \]

It is straightforward to verify that this value function is well-defined, as the maximum \( x^i \) is achieved, and fulfills the following properties: (i) it is bounded; (ii) it is concave; (iii) it is weakly increasing in \( v^i \) for every \( f^i \) and upper semi-continuous on the restricted domain \( D^i \); (iv) for every \( f^i \) in \( F^i \), it is continuous on the restricted domain \( \{(w^i, f^i) \in D^i : f^i = \bar{f}^i\} \)

and upper semi-continuous on the restricted domain \( \{(w^i, f^i) \in D^i : f^i \leq \bar{f}^i\} \); (v) for every \( f^i \) in \( F^i \) \( (x^i, v^i) \), by construction, \( (v^i, f^i) \) is an element of the domain \( D^i \) and, by the previous argument for optimality, at every date-event \( \sigma \) in \( S \),
\[ J^i_\sigma (v^i; f^i) = U^i_\sigma (x^i). \]

We now show some properties of differentiability of the value function. Given any \( f^i \) in \( F^i \) \( (x^i, v^i) \) and any \( w^i \) in \( V^i \) satisfying \( w^i \geq v^i - x^i \), \( (w^i, f^i) \) is an element of the domain \( D^i \) and, by optimality, at every date-event \( \sigma \) in \( S \),
\[
J^i_\sigma (w^i; f^i) \geq w^i (x^i + (w^i - v^i)) + \frac{1}{\pi^i_\sigma} \sum_{\tau \in \sigma^+} \pi^i_\tau J^i_\tau (v^i; f^i)
\]
\[
\geq w^i (x^i + (w^i - v^i)) + \frac{1}{\pi^i_\sigma} \sum_{\tau \in \sigma^+} \pi^i_\tau U^i_\tau (x^i).
\]

By a well-known result in convex analysis (see Benveniste and Sheinkman [4]), given any \( f^i \) in \( F^i \) \( (x^i, v^i) \), the value function admits a (partial) derivative at \( (v^i; f^i) \) in \( D^i \) and, at every date-event \( \sigma \) in \( S \),
\[ \partial J^i_\sigma (v^i; f^i) = \partial u^i (x^i) \]

Thus, given any \( f^i \) in \( F^i \) \( (x^i, v^i) \), consider any \( (w^i, f^i) \) in \( D^i \) satisfying, at some date-event \( \sigma \) in \( S \), \( J^i_\sigma (w^i; f^i) \geq U^i_\sigma (e^i) \) and \( w^i \leq v^i \). Concavity delivers
\[ U^i_\sigma (e^i) - U^i_\sigma (x^i) \leq J^i_\sigma (w^i; f^i) - J^i_\sigma (v^i; f^i) \leq \partial u^i (x^i) (w^i - v^i) \leq \xi (w^i - v^i), \]

where
\[ \xi = \bigwedge_{\sigma \in S} \partial u^i (x^i) > 0. \]

Therefore, rearranging terms,
\[ v^i - \frac{U^i_\sigma (x^i) - U^i_\sigma (e^i)}{\xi} \leq w^i. \]
Also, by uniform impatience and boundedness of per-period utility, there exists a sufficiently large $\phi > 0$ satisfying

$$
\phi > \sqrt[\sigma \in S]{\frac{U^i_\sigma(x^i) - U^i_\sigma(e^i)}{\xi}}.
$$

It follows that, given any $f^i$ in $F^i(x^i, v^i)$, for every $(w^i, f^i)$ in $D^i$,

(\dagger)

$$
J^i_\sigma(w^i_\sigma; f^i) \geq U^i_\sigma(e^i)
$$

only if $w^i_\sigma \geq v^i_\sigma - \phi$.

We shall exploit this fundamental inequality in order to recover not-too-tight debt limits.

We implicitly define an operator $G^i : F^i(x^i, v^i) \rightarrow F^i(x^i, v^i)$ by setting, at every date-event $\sigma$ in $S$,

$$
J^i_\sigma(-G^i(f^i)_\sigma; f^i) = U^i_\sigma(e^i).
$$

To prove that this operator is well defined, pegging any date-event $\sigma$ in $S$, observe that

$$
J^i_\sigma(0; f^i) \geq J^i_\sigma(0; 0) \geq U^i_\sigma(e^i)
$$

and

$$
J^i_\sigma(-g^i; f^i) \leq u^i(0) + \frac{1}{\bar{\pi}^i_\sigma} \sum \pi^i_\tau U^i_\tau(x^i) < U^i_\sigma(e^i),
$$

where

$$
0 \leq g^i_\sigma = \sup \{-w^i_\sigma \in \mathbb{R} : B^i_\sigma(w^i_\sigma; f^i) \text{ is non-empty}\} \leq c^i_\sigma + \frac{1}{p^i_\sigma} \sum p^i_\tau f^i_\tau,
$$

as utility satisfies boundary conditions. Hence, by the Intermediate Value Theorem, $G^i(f^i)$ exists in $L$. Also, it is positive and bounded, as market clearing and first-order conditions (c-1)-(c-2) imply

$$
\frac{1}{p^i_\sigma} \sum p^i_\tau f^i_\tau \leq \frac{1}{p^i_\sigma} \sum p^i_\tau \sum f^j_\tau = \frac{1}{p^i_\sigma} \sum p^i_\tau \sum (v^j_\tau + f^j_\tau)
$$

$$
= \sum \frac{1}{p^i_\tau} \sum p^i_\tau (v^j_\tau + f^j_\tau)
$$

$$
= \sum \frac{1}{p^i_\tau} \sum p^j_\tau (v^j_\tau + f^j_\tau)
$$

$$
\leq \sum \left( \bigvee_{\tau \in S} |v^j_\tau + f^j_\tau| \right) \frac{1}{p^i_\tau} \sum p^j_\tau
$$

and uniform impatience yields

$$
\frac{1}{p^i_\sigma} \sum p^i_\tau f^i_\tau \leq \frac{1}{\bar{\pi}^i_\sigma} \sum \pi^i_\tau \left( \frac{\partial w^i(\epsilon)}{\partial w(1/\epsilon)} \right) \leq \frac{1}{\eta} \left( \frac{\partial w^i(\epsilon)}{\partial w(1/\epsilon)} \right).
$$

Furthermore, at every date-event $\sigma$ in $S$,

$$
J^i_\sigma(v^i_\sigma; f^i) = U^i_\sigma(x^i) \geq U^i_\sigma(e^i) = J^i_\sigma(-G^i(f^i)_\sigma; f^i)
$$
and, at every date-event $\sigma$ in $S^i$,
\[
J^i_\sigma(v^i; f^i) = U^i_\sigma(x^i) = U^i_\sigma(e^i) = J^i_\sigma(-G^i_{f^i}\sigma; f^i).
\]
Hence, $G^i(f^i)$ is an element of $F^i(x^i, v^i)$. Finally, observe that the operator $G^i : F^i(x^i, v^i) \to F^i(x^i, v^i)$ is (weakly) monotone.

Construct debt limits $\bar{f}^i$ in $F^i(x^i, v^i)$ so that $\bar{f}^i_\sigma = -v^i_\sigma$, at every date-event $\sigma$ in $S^i$, and $\bar{f}^i_\sigma \geq -v^i_\sigma + \phi$, at every date-event $\sigma$ in $(S/S^i)$. We claim that $G^i(\bar{f}^i)$ in $F^i(x^i, v^i)$ satisfies $G^i(\bar{f}^i) \leq \bar{f}^i$. Indeed, exploiting restriction (†), at every date-event $\sigma$ in $(S/S^i)$,
\[
G^i(\bar{f}^i) \leq -v^i_\sigma + \phi \leq \bar{f}^i.
\]
at every date-event $\sigma$ in $S^i$,
\[
G^i(\bar{f}^i) = -v^i_\sigma = \bar{f}^i.
\]

Now, by induction, construct a sequence $((G^i)^n(\bar{f}^i))_{n \in T}$ in $F^i(x^i, v^i)$. Such a sequence is weakly decreasing and bounded, as
\[
\bar{f}^i \geq (G^i)^n(\bar{f}^i) \geq (G^i)^{n+1}(\bar{f}^i) \geq -v^i.
\]
Hence, it converges to some $f^i$ in $F^i(x^i, v^i)$ in the product topology. By upper semi-continuity of the value function, at every date-event $\sigma$ in $S$,
\[
J^i_\sigma(-f^i_\sigma; f^i) \geq U^i_\sigma(e^i).
\]
Finally, suppose the latest holds with strict inequality at some date-event $\sigma$ in $S$, i.e., there exists $\epsilon > 0$ such that
\[
J^i_\sigma(-f^i_\sigma; f^i) > J^i_\sigma(-f^i_\sigma - \epsilon; f^i) > U^i_\sigma(e^i).
\]
For every sufficiently large $n$ in $T$, $(G^i)^{n+1}(\bar{f}^i) \leq f^i_\sigma + \epsilon$ and, therefore,
\[
U^i_\sigma(e^i) \geq J^i_\sigma(-(G^i)^{n+1}(\bar{f}^i); (G^i)^n(\bar{f}^i)) \\
\geq J^i_\sigma(-(f^i_\sigma - \epsilon; (G^i)^n(\bar{f}^i)) \\
\geq J^i_\sigma(-(f^i_\sigma - \epsilon; f^i)) \\
> U^i_\sigma(e^i),
\]
a contradiction. Hence, $f^i$ in $F^i$ are not-too-tight debt limits at equilibrium. □

The proof is now complete. □

**Proof of lemma 4.** Let $M(\epsilon)$ be the set of Malinvaud efficient allocations. The proof of the Second Welfare Theorem implicitly establishes the existence of a correspondence $G : M(\epsilon) \rightarrow \Omega$ giving distributions of initial claims which are consistent with a given equilibrium allocation. Such a correspondence is closed with non-empty convex values. In addition, at no loss of generality, for some sufficiently large $\mu > 0$, we can truncate $\Omega$ as
\[
\Gamma = \left\{ \omega \in \Omega : \sum_{i \in J} |\omega^i| \leq \mu \right\},
\]
which is a compact convex set. Indeed, we identify uniform upper bounds for initial claims in the part on recovering financial plans of the proof of proposition 3.
Hence, by the Closed Graph Theorem, the correspondence $G : M ( e ) \rightarrow \Gamma$ is upper hemi-continuous with non-empty convex values.

Let $f_\theta ( e, y ) = \arg \max_{z \in F ( e, y )} W_\theta ( z )$.

By Berge’s Maximum Theorem, the correspondence $f ( e, y ) : \Theta \mapsto F ( e, y )$ is upper hemi-continuous with non-empty values. Suppose that $\{ x, z \} \subset f_\theta ( e, y )$. For every $1 > \lambda > 0$, allocation $\lambda x + (1 - \lambda) z$ lies in $F ( e, y )$. By convexity of preferences, it follows that

$$\sum_{i \in J} \theta^i \left[ U^i ( \lambda x^i + (1 - \lambda) z^i ) - \lambda U^i ( x^i ) - (1 - \lambda) U^i ( z^i ) \right] = 0.$$ 

Thus, for every individual $i$ in $\{ i \in J : \theta^i > 0 \}$, $x^i = z^i$. Suppose that, for some individual $i$ in $\{ i \in J : \theta^i = 0 \}$, $x^i \neq z^i$. This implies that

$$U^i ( \lambda x^i + (1 - \lambda) z^i ) > \lambda U^i ( x^i ) + (1 - \lambda) U^i ( z^i ) \geq U^i ( e^i ).$$

As a slight reduction of initial consumption of this individual does not affect participation, and such resources can be distributed to individuals in $\{ i \in J : \theta^i > 0 \}$, this produces a contradiction. It follows that $f ( e, y ) : \Theta \mapsto F ( e, y )$ is an upper hemi-continuous correspondence with single values and, in fact, a continuous function. Hence, $M ( e, y )$ is connected, being the image of a connected set by a continuous function. Thus, $G ( M ( e, y ) )$ is connected, being the image of a connected set by an upper hemi-continuous correspondence with non-empty convex values in a topological vector space.

Construct now a correspondence $F : \Theta \times \Gamma \mapsto \Theta \times \Gamma$ by setting, at every $(\bar{\theta}, \bar{\omega})$ in $\Theta \times \Gamma$,

$$F_{\Theta} ( \bar{\theta}, \bar{\omega} ) = \arg \min_{\theta \in \Theta} \theta \cdot \bar{\omega}$$

and $F_{\Gamma} ( \bar{\theta}, \bar{\omega} ) = G ( f_\theta ( e, y ) )$.

This correspondence is upper hemi-continuous with non-empty convex values. By Kakutani’s Fixed Point Theorem, it admits a fixed point $(\bar{\theta}, \bar{\omega})$ in $\Theta \times \Gamma$.

Suppose that, at the fixed point, for some individual $i$ in $J$, $\bar{\omega}^i > 0$. This implies that $\bar{\theta}^i = 0$ and, by optimality, $U^i \left( \frac{f_\theta ( e, y )^i}{n} \right) = U^i ( e )$. This is a contradiction, as a strictly positive initial claim permits to achieve a utility strictly higher than the autarchic utility, subject to budget feasibility, at positive debt limits. Hence, $\bar{\omega} = 0$, which proves the claim. \(\square\)

**Proof of proposition 4.** Given strictly positive welfare weights $\theta$ in $\Theta$, we exploit proposition 1 to peg any sequence of allocations $(y^n)_{n \in \mathbb{N}}$ in $X ( e )$ satisfying, at every $n$ in $\mathbb{N}$,

$$W_\theta ( y^n ) = W_\theta ( e ) + \left( \frac{1}{n} \right) \left( \max_{z \in X ( e )} W_\theta ( z ) - W_\theta ( e ) \right)$$

and

$$W_\theta ( y^n ) = \max_{z \in F ( e, y^n )} W_\theta ( z ).$$

Notice that this sequence consists of distinct non-autarchic allocations. Applying lemma 4, there exists a sequence of non-autarchic allocations $(x^n)_{n \in \mathbb{N}}$ in $E_\theta ( e )$ such that, for every $n$ in $\mathbb{N}$, $x^n$ lies in $M ( e, y^n )$. Furthermore, by construction, for every $n$ in $\mathbb{N}$,

$$W_\theta ( x^n ) \leq W_\theta ( e ) + \left( \frac{1}{n} \right) \left( \max_{z \in X ( e )} W_\theta ( z ) - W_\theta ( e ) \right).$$
It follows that the sequence \((x^n)_{n \in \mathbb{N}}\) in \(E_0(e)\) consists of infinitely many distinct allocations, for otherwise some subsequence would converge to an allocation \(x\) in \(X(e)\) with \(W_\theta(x) > W_\theta(e)\), violating the above inequality. \(\Box\)

**Appendix B. Severe punishments**

By reducing the risk of default, severe punishments enlarge insurance opportunities and increase social welfare at equilibrium. More relevantly for our analysis, they guarantee constrained efficiency of competitive equilibrium and, thus, eliminate indeterminacy.

Assume that the punishment for default consists of the exclusion from financial markets and, in addition, a partial confiscation of private endowment. This hypothesis can be accommodated in our analysis by modifying participation for constrained efficiency and by adapting restrictions for not-too-tight debt limits at equilibrium. In particular, let \(1 > \lambda \geq 0\) represent the portion of unexpropriable private endowment after debt repudiation. For every individual \(i\) in \(J\), at every date-event \(\sigma\) in \(S\), participation now imposes

\[
U^i_\sigma(x^i) \geq U^i_\sigma(\lambda e^i).
\]

Furthermore, at a competitive equilibrium, debt limits satisfy

\[
J^i_\sigma(-f^i_\sigma; f^i) = U^i_\sigma(\lambda e^i).
\]

We argue that competitive equilibrium exhibits high implied interest rates \((i.e., a finite present value of intertemporal endowment), so guaranteeing that a constrained efficient allocation of risk is achieved (by a straightforward adaptation of Alvarez and Jermann’s [4, Corollary 4.7] First Welfare Theorem).

Suppose that, for some individual \(i\) in \(J\), at date-event \(\sigma\) in \(S\),

\[
p_\sigma f^i_\sigma < (1 - \lambda) p_\sigma e^i_\sigma + \sum_{\tau \in \sigma^+} p_\tau f^i_\tau.
\]

This means that, beginning with the maximum amount of debt and issuing future contingent debt up to the limit, a consumption level strictly above the unexpropriable endowment is affordable. Therefore, by strict monotonicity of preferences,

\[
J^i_\sigma(-f^i_\sigma; f^i) > u^i(\lambda e^i_\sigma) + \frac{1}{p_\sigma} \sum_{\tau \in \sigma^+} \pi^i_\tau J^i_\tau(-f^i_\tau; f^i) = u^i(\lambda e^i_\sigma) + \frac{1}{p_\sigma} \sum_{\tau \in \sigma^+} \pi^i_\tau U^i_\tau(\lambda e^i) = U^i_\sigma(\lambda e^i),
\]

which is a contradiction. Hence, for every individual \(i\) in \(J\), at every date-event \(\sigma\) in \(S\),

\[
p_\sigma f^i_\sigma \geq (1 - \lambda) p_\sigma e^i_\sigma + \sum_{\tau \in \sigma^+} p_\tau f^i_\tau.
\]

Consolidating over date-events, and exploiting positivity of debt limits,

\[
f^i_\sigma \geq (1 - \lambda) \frac{1}{p_\sigma} \sum_{\tau \in \sigma^+(\sigma)} p_\tau e^i_\tau.
\]

This shows that, at equilibrium, prices involve high implied interest rates.
Incidentally, notice that, when the entire endowment can be confiscated after default ($\lambda = 0$), for every individual $i$ in $J$, at every date-event $\sigma$ in $S$, debt limits satisfy

$$p_\sigma f^i_\sigma = p_\sigma e^i_\sigma + \sum_{\tau \in \sigma_+} p_\tau f^i_\tau.$$ 

Thus, by consolidation,

$$f^i_\sigma = \frac{1}{p_\sigma} \sum_{\tau \in S(\sigma)} p_\tau e^i_\tau + b^i_\sigma,$$

where $b^i$ is a positive element of $L$ satisfying the martingale property

$$p_\sigma b^i_\sigma = \sum_{\tau \in \sigma_+} p_\tau b^i_\tau.$$

This bubble component vanishes at equilibrium by canonical arguments. Hence, natural debt limits (that is, debt is bounded by the present value of private endowment) correspond to not-too-tight debt limits with full confiscation of private endowment after default.

**Appendix C. Determinacy example**

Within the context of the example in section 2, we now perform the following exercise. We peg the initial claim corresponding to the steady state equilibrium with trade,

$$v^* = \left( 1 + \frac{\beta u'(e_l + \xi^*)}{u'(e_h - \xi^*)} \right)^{-1} \xi^*.$$ 

For every allocation $\xi_0$ in $(0, \xi^*)$, we determine the sequence of claims $(v_t)_{t \in T}$ consistent with the initial value $v_0 = v^*$. We then verify whether such claims remain positive and, thus, whether they sustain a competitive equilibrium under positive debt limits. Positivity is hard to ascertain by means of analytical methods and, therefore, we rely on numerical computations, although these cannot be decisive on this matter.

Endowments are $(e_l, e_h) = (0.7, 1.3)$. The discount factor is $\beta = .8$, whereas the utility function is given by $u(c) = \sqrt{c}$. We find that, after an initial phase, claims (and, hence, debt limits) oscillates between positive and negative values, over some range, before converging to zero. These negative values of debt limits are smaller, and occur in a farer horizon, the closer is the initial condition $\xi_0$ to efficiency $\xi^*$. Our grid of initial values $\xi_0$ is contained in $(0, \xi^*)$ with increments of $10^{-6}$. The four panels below report the sequences of claims for different initial conditions $\xi_0$ in $(0, \xi^*)$.

These computations cast some doubts on the claim of indeterminacy submitted by Azariadis, Antinolfi and Bullard [6] (see, also, Azariadis [7]). It seems that the constrained efficient steady state is determinate and, indeed, the unique equilibrium of an economy with that distribution of initial claims, when debt limits are restricted to be positive. As previously mentioned, all equilibria are indeterminate when debt limits are not constrained by positivity.
(a) $\xi_0 = \xi^* - 10^{-6}$

(b) $\xi_0 = \frac{2}{3} \xi^*$

(A) $\xi_0 = \frac{1}{3} \xi^*$

(B) $\xi_0 = 10^{-6}$