8. THE MARKET ECONOMY, COMPETITIVE TRADE AND EQUILIBRIUM

In this section we consider two economies differing by markets arrangement. The first is an extension of the Walrasian (or Arrow-Debreu) economy in which agents can both trade on spot markets and over a complete set of future markets. Thus, at every date an agent can also buy and sell a commodity to be delivered in any future date. The second economy is more realistic as it only allows for sequential trade: at all dates, agents can only trade on a spot market for the commodity and on a bond market. Therefore, they can transfer wealth (and real resources) over time only by buying and issuing (i.e. saving and borrowing in) the bond. We then show that, under appropriate conditions, both economies deliver equilibrium allocations that are Pareto efficient (i.e. the I Welfare Theorem holds) and, conversely, that any Pareto optimal allocation can be attained as a competitive equilibrium of one of the two economies (i.e. the II Welfare Theorem holds).

The II Welfare Theorem is a particularly relevant result, since it is the formal justification for directly using the central planner problem to analyze issues such as growth and business cycle, as we have been doing so far.

8.1. The Walrasian economy. We distinguish between spot prices and present value prices. In each date \( t \) there are three spot markets, for the commodity, labor, capital. We denote spot prices in units of the commodity (identified as the numéraire), respectively, as \((1, w_t, \nu_t)\). Next, we denote by \( p_t \) the future or forward price of a unit of commodity in date \( t \) as quoted in the initial date 0; \( p_t \) is also called present value or Arrow-Debreu price.\[^33\]

Notice that, given an entire sequence of present value prices \((1, p_1, p_2, ..., p_t, ...)\) we can evaluate future prices at any non-initial date; for example, the price of a unit of consumption in \( T > t \) quoted in \( t \) is \( p_T / p_t \). Finally, for two consecutive dates, \( t, t + 1, \) \( p_{t+1} / p_t \) is the value of a unit of commodity in \( t + 1 \) quoted at \( t \), that is discounted back at \( t \). Accordingly, we can implicitly define a discount rate as the value \( r_t \) satisfying,

\[
\frac{p_{t+1}}{p_t} = \frac{1}{1 + r_t}
\]

To summarize, we have two equivalent formulations of the Walrasian economy, one that exploits present value prices, and another, often used in macro textbooks, that makes use of discount factors.

8.1.1. Firms. A plan \( y = (y_0, y_1, ..., y_t, ...) \) is short-run profit maximizing if at all \( t, y_t \in Y \) solves,

\[
\pi_t = \max (\gamma_t \cdot y_t),
\]

at given spot markets price \( \gamma_t \). Thus, the present value of the intertemporal profits, evaluated at time zero, is

\[
\Pi_0 = \max \sum_{t \in T} p_t (\gamma_t \cdot y_t).
\]

In the Ramsey economy, we explicit this problem for the following two cases.

[Case 1] A firm has to rent out capital on the market;
[Case 2] A firm owns capital and decides about capital accumulation.

The main difference between these two cases is that capital is traded only in the first; thus the first will corresponds to an economy with three markets and the second to one with two markets only.

Case 1. In this case $y_t = (y_{1,t}, y_{2,t}, y_{3,t})$ corresponds to output to be offered for consumption, labor input and capital input, $(c_t, -n_t, -k_t)$. To simplify notation, we write the production possibility set directly in terms of the latter,

$$Y = \{ (c_t, -n_t, -k_t) : c_t \leq F(k_t, n_t), 0 \leq n_t \leq 1, k_t \geq 0 \} - \mathbb{R}_+^3$$

Letting $\gamma_t := (1, w_t, \nu_t)$, the short run profit function is,

$$\pi_t = \max \left[ F(k_t, n_t) - w_t n_t - \nu k_t \right], \quad 0 \leq n_t \leq 1, k_t \geq 0.$$

The present value of the intertemporal profits, evaluated at time zero, is

$$\Pi_0 = \max \sum_{t \in T} p_t \left[ F(k_t, n_t) - w_t n_t - \nu k_t \right]$$

Case 2. Now, suppose the firm owns and accumulates capital. Thus, in the representative firm economy, there is no market for capital. In this case, we let $y_t := (c_t, -n_t)$, and define the production possibility set as,

$$Y = \left\{ (c_t, -n_t) : c_t + i_t \leq F(k_t, n_t), 0 \leq n_t \leq 1, k_{t+1} = i_t + k_t (1 - \delta), k_{t+1}, i_t \geq 0, k_t \geq 0 \text{ given} \right\} - \mathbb{R}_+^2$$

Letting spot prices be $\gamma_t = (1, w_t)$, the short run profit function is,

$$\pi_t = \max [c_t - w_t n_t], \quad (c_t, -n_t) \in Y$$

or

$$\pi_0 = \max F(k_t, n_t) + k_t (1 - \delta) - k_{t+1} - w_t n_t], \quad 0 \leq n_t \leq 1, k_{t+1} \geq 0, k_t \geq 0 \text{ given}.$$

The present value of the intertemporal profits, evaluated at time zero, is

$$\Pi_0 = \max \sum_{t \in T} p_t \left[ F(k_t, n_t) + k_t (1 - \delta) - k_{t+1} - w_t n_t \right]$$

Notice that in case 1 the firm problem is essentially static, while in case two it is dynamic: the current, individual investment decision affects future production possibilities and decisions.

8.1.2. The household. In the initial date the household maximizes,

$$\sum_{t \in T} \beta^t u(c_t, n_t)$$

over her budget set. The budget set differs in the two cases above, depending on whether or not the household owns and rents out capital. Assuming there is no (exogenous) endowment of the commodity and that the household is the firm owner (earning profits), we have the following.
Case 1. Given \((p, w, \nu)\), \(\pi\) and \(k_0\), the household chooses \((c_t, n_t, i_t)_{t \in T}\) such that,
\[
\sum_{t \in T} p_t [i_t + c_t - w_t n_t - \nu_t k_t - \pi_t] \leq 0,
\]
or chooses \((c_t, n_t, k_{t+1})_{t \in T}\) such that,
\[
\sum_{t \in T} p_t [c_t - c_{t+1} - k_t (1 - \delta) - w_t n_t - \nu_t k_t - \pi_t] \leq 0.
\]

Case 2. Given \((p, w)\) and \(\pi\), the household chooses \((c_t, n_t)_{t \in T}\) such that,
\[
\sum_{t \in T} p_t [c_t - w_t n_t - \pi_t] \leq 0
\]
Denoting by \(\omega_t\) the real ‘disposable’ income (i.e. the income available for consumption) at \(t\), the intertemporal budget constraint in the initial date can be written as,
\[
\sum_{t \in T} p_t (c_t - \omega_t) \leq 0
\]
with
\[
(\omega) \quad \omega_t := (-k_{t+1} + k_t (1 - \delta) + w_t n_t + \nu_t k_t + \pi_t), \quad \text{or} \quad \omega_t := (w_t n_t + \pi_t).
\]
respectively, in case 1 and 2.

An important property of individually optimal allocations follows.

**Lemma 8.** Let \((c, n)\), be an individually optimal allocation at \((\omega, p)\), then any \((c', n')\) that is strictly preferred to \((c, n)\) will not be budget-feasible at \(p\).

The proof is easy, since if \((c', n')\) where budget feasible it would contradict the individual optimality of \((c, n)\). This lemma holds provided that preferences satisfy local-non-satiation, a weaker requirement than strict monotonicity; in fact, this implies that, at an individual optimum, the household spends all her lifetime income (i.e. her intertemporal, budget constraint holds with equality).\(^{34}\)

8.1.3. *Walrasian equilibrium.* An economy is specified by \(E = (U, Y, k_0)\).

**Definition 9 (Walrasian Equilibrium).** A Walrasian equilibrium of an economy \(E\) is an allocation \((c, n, i, y)\) and prices \((p, \gamma)\) that
- solves the household problem at \((p, \gamma)\) and \(\pi = (\gamma_t \cdot y_t)_{t \in T}\),
- solves the firm intertemporal profit maximization at \((p, \gamma)\),
- satisfies spot-t markets clearing at all dates \(t\) in \(T\).

The market clearing conditions vary depending on which markets are available. If the firm rents capital (Case 1 above) there are three spot markets, the one for the consumption commodity, the labor market and the capital market. Thus, at every date \(t\) in \(T\), given \(i_t = -y_{3,t+1} + (1 - \delta)y_{3,t}\) and, by market clearing, \(c_t + i_t - F(k_t, y_{2,t}) = 0\), \(y_{2,t} + n_t = 0\) and \(y_{3,t} + k_t = 0\). If, as in case 2, the firm owns capital and we assume there is no trade for capital, only the first two market clearing conditions have to hold at equilibrium.

\(^{34}\)See, for example, Mas-Colell Whiston and Green. Microeconomic Theory. 1995.
Finally, let us derive Walras law. Walras law holds if and only if, at an equilibrium, the present value of excess demands equal the present value of excess supplies. In our context, this implies,

\[(WL) \quad \sum_{t \in T} p_t [c_t + i_t - F(k_t, n_t)] = 0\]

To understand the latest, observe that, because preferences are strictly increasing, every individually optimal consumption allocation satisfies the household’s budget constraint with equality.

\[\sum_{t \in T} p_t (c_t - \omega_t) = 0\]

Both in case 1 and 2 above, at equilibrium,

\[\gamma, c_t - \omega_t = c_t + i_t - F(k_t, n_t), \quad i_t = y_{3,t+1} - y_{3,t}(1 - \delta), \quad y = (y_1, y_2, y_3) = (c_t, -n_t, -k_t)_{t \in T}\]

Substituting for \(\omega\) in the budget constraint, yields (WL).

By Walras law, we can drop one market clearing condition: at each \(t\) in \(T\), if all market clearing conditions but one hold, also the remaining one will be satisfied. Conceptually, this says that, at each \(t\) in \(T\), an equilibrium does only allow to pin down relative prices, which, under our price normalization, are quoted in terms of the, date zero, consumption good.

8.1.4. Equilibrium properties.

**Theorem 9 (I Welfare Theorem).** Every Walrasian equilibrium is Pareto efficient.

The proof of this theorem is standard, slightly simpler because of the representative agent assumption.

**Proof.** By contradiction, suppose the Walrasian equilibrium, with allocation \((c, n, i, y)\) and prices \((p, \gamma)\), of a given economy \(E\), is not Pareto optimal. Hence, there is an alternative allocation \((c', n', i', y')\) that is resource-feasible, namely at all \(t\) in \(T\),

\[c_t + i_t - F(k_t, n_t) \leq 0,\]

\[y_t = (y_{1,t}, y_{2,t}, y_{3,t}) = (c'_t, -n'_t, -k'_t), \quad i'_t = -y_{3,t+1}' + y_{3,t}(1 - \delta) = 0, \quad k_0 = k_0',\]

and it is welfare improving, \(U(c', y'_2) > U(c, y_2)\). Using \((\omega)\) and \((\pi'_t)_{t \in T} = (\gamma_t \cdot y'_t)_{t \in T}\), define \(\omega'\) so as to be compatible with the alternative allocation \((c', n', i', y')\) at \((p, \gamma)\). By lemma 8,

\[\sum_{t \in T} p_t (c'_t - \omega'_t) > 0\]

Since, \(c'_t - \omega'_t \leq c'_t + i'_t - F(k'_t, n'_t)\), the latest inequality reads,

\[\sum_{t \in T} p_t [c'_t + i'_t - F(k'_t, n'_t)] > 0\]

As the present value prices are nonnegative (strictly positive by strict monotonicity of preferences), the latest implies that

\[c'_t + i'_t - F(k'_t, n'_t) > 0, \quad at \ some \ t \ in \ T\]

\[\text{35 Indeed, by individual optimality, } p_T = \text{proportional to } \beta^T u_c(c_t, n_t) \text{ (the coefficient of proportionality is the positive scalar determined by the Lagrange multiplier associated to the intertemporal budget constraint).}\]
which contradicts resource-feasibility of \((c', n', i', y')\).

It is also instructive to verify the properties of an interior equilibrium allocation.

**Proposition 10.** Every interior Walrasian equilibrium allocation satisfies the Euler equation, the optimal labor supply condition and the transversality condition.

This, by proposition 1, also provides an alternative proof of the I Welfare Theorem.

**Proof.** First, form the Lagrangian for the consumer problem. The maximum of this problem satisfies first order conditions: at all \(t\) in \(T\),

\[
\frac{u_c(c_t, n_t)}{\beta u_c(c_{t+1}, n_{t+1})} = \frac{p_t}{p_{t+1}}
\]

\[
- \frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t
\]

\[
\frac{p_t}{p_{t+1}} = 1 - \delta + \nu_t
\]

Then, compute the first order conditions for the firm individual optimum and verify that the following hold,

\[
\nu_t = F_k(k_t, n_t), \quad w_t = F_n(k_t, n_t)
\]

Combining the two set of conditions yields the result,

\[
\frac{u_c(c_t, n_t)}{\beta u_c(c_{t+1}, n_{t+1})} = \frac{p_t}{p_{t+1}} = 1 - \delta + F_k(k_t, n_t),
\]

\[
- \frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t = F_n(k_t, n_t)
\]

where \(1 - \delta + F_k(k_t, n_t)\) defines the gross rate of interest \(1 + r_t\), and

\[
r_t = F_k(k_t, n_t) - \delta
\]

the interest rate.

Next, let us verify that \((T)\) holds. Observe that since preferences are strictly increasing in consumption, at an equilibrium, the fact that the household problem has a solution implies that the present value of her lifetime wealth is bounded, yielding,

\[
\sum_{t \in T} p_t \omega_t < \infty,
\]

Thus,

\[
\sum_{t \in T} p_t c_t < \infty
\]

As the consumption sequence is interior (strictly positive at each date) and prices are nonnegative, the latest implies that prices vanish to zero over time, ensuring that the transversality condition \((T)\) holds,

\[
\lim_{T \to \infty} p_T k_{T+1} = 0
\]
Next, it follows from the uniqueness of Pareto efficient allocation, in proposition 3, that,

**Proposition 11** (Determinacy). *Every Walrasian economy has a unique equilibrium.*

These properties, efficiency and determinacy, extend to economies with heterogeneous households and firms. The approach is due to Negishi’s work at the end of the ’50s.

8.2. **An economy with sequential trade.** The economy with sequential trade is like the Walrasian economy except that it does not have future or forward markets. Agents cannot trade today for a commodity (good or service) to be delivered in the future. However, they can transfer income over time using a one-period bond, which they can purchase and sell in every period, competitively, at the market price $q_t$.

8.2.1. *The household.* Let $a_t$ denote the household’s bond position at the beginning of period $t$, negative in case it records a liability/debt contracted in the previous date. Accordingly, $a_{t+1}$ denotes the bond position built up in period $t$, to be held at the beginning of period $t + 1$. The bond pays out a unit of commodity in every period. Thus, the household, date $t$, sequential budget constraint imposes that the current consumption $c_t$ and bond potion $a_{t+1}$ satisfies,

$$q_t a_{t+1} + c_t - ω_t ≤ a_t$$

at price $q_t$, real income $ω_t$ and initial bond position $a_t$. Obviously, real income vary depending on whether we consider case 1 or case 2 above, respectively,

$$ω_t := (-k_{t+1} + k_t(1 - δ) + w_t n_t + ν_t k_t + π_t), \text{ or } ω_t := (w_t n_t + π_t).$$

With strictly increasing preferences over consumption ($u' > 0$), the household problem is well defined, and has a solution, provided one imposes some limit on the amount of debt she can issue. With no debt limits, the household is left free to expand consumption arbitrarily at any date without having to modify consumption in any other date, simply financing the extra expenditure in $t$ by issuing the necessary amount of debt and rolling it over thereafter. Such a strategy is called Ponzi scheme (or Ponzi game). A sufficient condition to rule out Ponzi schemes is to assume that bond trading strategies fulfill condition,

\[(NPG) \liminf_{T \to \infty} \left( \prod_{t=0}^{T} q_t \right) a_T ≥ 0.\]

that is, essentially, no debt can be carried out indefinitely. Intuitively, this excludes debt growing at a rate higher than the market interest rate. To see this, suppose the interest rate $(1 + r) = 1/q$ is constant over time, and assume debt raised in date $t$ grows at a constant rate $0 ≤ α < 1$, $a_{t+1} = (1 + α)a_t$.\footnote{T.J. Kehoe, D.K. Levine and P.M. Romer (1990). Determinacy of equilibria in dynamic models with finitely many consumers, *Journal of Economic Theory*, 50(1), 1-21.} After $T + 1$ periods, $a_{t+1+T} = (1 + α)^{T+1}a_t$. Substituting in
the left hand side of (NPG),

$$\lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^{T+1} (1 + \alpha)T+1a_t = a_t \lim_{T \to \infty} \left( \frac{1 + \alpha}{1 + r} \right)^{T+1}$$

the latest is nonnegative (actually, zero) if and only if \( \alpha < r \), so that the term in brackets is less than one, collapsing to zero in the limit.

An alternative, equivalent, formulation of the NPG condition can be expressed using (implicit) present value prices. To see this, we are going to construct present value prices from bond prices. Indeed, observe that \( q_t \) defines the price in \( t \) of a unit of consumption delivered in \( t + 1 \). Thus,

$$q_t = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_t}$$

This implies that,

$$q_0 = p_1$$

$$q_0 q_1 = \frac{p_1 p_2}{p_1} = p_2$$

$$q_0 q_1 q_3 = \frac{p_1 p_2 p_3}{p_1 p_2} = p_3$$

$$\cdots \cdots \cdots$$

$$\prod_{t=0}^T q_t = p_T$$

and allows to write,

(NPG) \( \lim_{T \to \infty} p_T a_T \geq 0 \).

To summarize, at all \( t \) in \( T \), given \((a_t, p)\), the household problem is defined as,

$$\max_{(c_t, a_t)} \sum_{s \in T} \beta^s u(c_{t+s}, n_{t+s})$$

$$\frac{p_{t+1+s}}{p_{t+s}} a_{t+1+s} + c_{t+s} - \omega_{t+s} \leq a_{t+s},$$

$$\lim_{T \to \infty} \frac{p_T}{p_t} a_T \geq 0.$$

where, according to \((\omega)\), \( \omega \) is a function of \( n \), and (NPG) is imposed.

In order to characterize the solution of this problem, we prove the following.

Proposition 12. An interior allocation \((c, n, a)\) solves the household problem at \((a_0, p)\) if, at all \( t \) in \( T \), the following conditions hold,

- \((E')\) \( \frac{u_c(c_t, n_t)}{\beta u_c(c_{t+1}, n_{t+1})} = \frac{p_t}{p_{t+1}} \)
- \((L')\) \( \frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t \)
- \((T')\) \( \limsup_{T} p_T a_T \leq 0 \)
Observe that \((E', L', T')\) respectively denote the Euler equation, the labor supply and the transversality condition.

**Proof.** By contradiction, suppose \((c, n, a)\) satisfies \((E'), (L'), (T')\) but it is not individually optimal. Then there exists an alternative plan \((c', n', a')\), with \(a_0 = a'_0\), that is budget feasible, satisfies the NPG condition and yields an higher lifetime utility. At any \(t\) in \(T\), feasibility of the alternative plan and optimality of the original plan, respectively imply,

\[
p_{t+1}a'_{t+1} + p_t c'_{t+1} - p_t \omega'_t \leq p_t a'_t
\]
\[
p_{t+1}a_{t+1} + p_t c_t - p_t \omega_t = p_t a_t
\]

Subtracting the second from the first,

\[
p_t (c'_t - c_t) - p_t (\omega'_t - \omega_t) \leq -p_{t+1} (a'_{t+1} - a_{t+1}) + p_t (a'_t - a_t)
\]

Summing over \(t = 0, 1, ..., T\) and simplifying,

\[
\sum_{t=0}^{T} p_t \left[ (c'_t - c_t) - (\omega'_t - \omega_t) \right] \leq -p_{T+1} (a'_{T+1} - a_{T+1}) + p_T (a'_0 - a_0)
\]

Suppose, as in case 2, that \(\omega_t := w_t n_t + \pi_t\) and \(\omega'_t := w_t n'_t + \pi_t\) (\(\pi_t\) is taken as given by the household).

\[
\sum_{t=0}^{T} p_t \left[ (c'_t - c_t) - w_t (n'_t - n_t) \right] \leq -p_{T+1} (a'_{T+1} - a_{T+1})
\]

Take the lim sup,

\[
(+) \limsup_{T} \sum_{t=0}^{T} p_t \left[ (c'_t - c_t) - w_t (n'_t - n_t) \right] \leq - \liminf_{T} p_{T+1} a'_{T+1} + \limsup_{T} p_{T+1} a_{T+1}
\]

\[
\leq - \liminf_{T} p_{T+1} a'_{T+1}
\]

\[
\leq (\text{NPG}) 0
\]

Next, exploiting the concavity of utility,

\[
U(c', n') - U(c, n) \leq \limsup_{T} \sum_{t=0}^{T} \beta^t \left[ u_c(c_t, n_t) (c'_t - c_t) + u_n(c_t, n_t) (n'_t - n_t) \right]
\]

\[
= \limsup_{T} \sum_{t=0}^{T} p_t \left[ (c'_t - c_t) - w_t (n'_t - n_t) \right]
\]

\[
\leq 0
\]

where in the second expression we have used first order optimality conditions, and the last inequality follows from \((+)\). Therefore, we attained the desired contradiction. \(\square\)
8.2.2. The firm. The firm sequentially solves its intertemporal maximization problem, discounting profits at the market interest rate or, equivalently at the implicit present value prices. As for the household, intertemporal income transfers can only be achieved, at each date, by trading the available bond.

At all $t$ in $\mathcal{T}$, the firm solves,

$$
\hat{\Pi}_t = \max_{(y_{t+s}, a_{t+1+s}) \in \mathcal{T}} \sum_{s \in \mathcal{T}} \frac{p_{t+s}}{p_t} \left[ \gamma_{t+s} \cdot y_{t+s} - q_{t+s}a^f_{t+1+s} + a^f_{t+s} \right], \quad y_{t+s} \in Y, \quad \text{at all } s \in \mathcal{T}
$$

given $a^f_t$ (a claim if $a^f_t > 0$ and a liability otherwise), denoting the bond position of the firm at the beginning of date $t$ and given prices $(q, \gamma)$. Similarly to the household, debt issue must be restricted in order $\hat{\Pi}_t$ to be well defined, at any date $t$ in $\mathcal{T}$. Thus, a similar no-Ponzi-game condition can be imposed on the financial policy of firm, in terms of equivalent, present value prices, at all $t$ in $\mathcal{T}$,

$$(\text{NPGf}) \quad \liminf_{T \to \infty} \frac{\Pi_T}{a^f_t} \geq 0.$$

Observe that at a solution of the firm problem,

$$q_t = \frac{p_{t+1}}{p_t}$$

Indeed, if it were not so, the firm would increase her (short-run) profits either by issuing the bond to be repaid in the next period (if $q_t > p_{t+1}/p_t$) or by buying it (if $q_t < p_{t+1}/p_t$). This says that solving the above problem is essentially equivalent to solve,

$$\Pi_t = \max_{(y_{t+s}) \in \mathcal{T}} \sum_{s \in \mathcal{T}} \frac{p_{t+s}}{p_t} \left[ \gamma_{t+s} \cdot y_{t+s} \right], \quad y_{t+s} \in Y, \quad \text{at all } s \in \mathcal{T}$$

Moreover, this equivalence can be interpreted by saying that, at an individual optimum, the financial policy of the firm is irrelevant, the so called 'Modigliani-Miller’s theorem’. For this result to be true, it must be that the firm does not face a binding borrowing constraint.

8.2.3. Sequential Competitive Equilibrium. Assuming there is no initial bond position ($a^f_0 = 0 = a_0$) the economy is again specified by $\mathcal{E} = (U, Y, k_0)$.

**Definition 10** (Sequential Competitive Equilibrium). A Sequential Competitive Equilibrium equilibrium of an economy $\mathcal{E}$ is an allocation $(c, n, i, y, a)$ and prices $(p, \gamma)$, that

- solve the household sequential problem at $(p, \gamma)$, given the firm profit $(\pi_t = \gamma_t \cdot y_t)_{t \in \mathcal{T}}$,
- solve the firm sequential profit maximization at $(p, \gamma)$,
- satisfy spot–$t$ markets clearing, at all $t$ in $\mathcal{T}$,
- satisfy bond market clearing, at all $t$ in $\mathcal{T}$.

Observe that in the equilibrium definition it suffices to impose spot markets clearing for labor and capital and use bond-market clearing condition to pin down the equilibrium present value prices. Once the latter are determined, one can derive the equilibrium bond prices and interest rates as previously explained.
8.2.4. Equilibrium properties. We are going to show that the no-Ponzi-game condition suffices to establish the equivalence between the individual optimality conditions in the the two economies considered, the Walrasian and the one with sequential markets. For the household this is done by proving that the household’s budget set in the two economies are identical, so that the sequential market arrangement does not provide less allocation opportunities than those offered in a Walrasian setting. This implies that every sequential equilibrium allocation is a Walrasian equilibrium allocation. Hence by theorem 16, every sequential equilibrium allocation is Pareto efficient, i.e. the I Welfare Theorem continues holds. For the same reason, we can also extend the determinacy result.

**Proposition 13.** Given $a_0$ and $p$, if a consumption plan $c$ satisfies the sequential budget constraint and (NPG), it also satisfies,
\[
\sum_{t=0}^{\infty} p_t(c_t - \omega_t) \leq a_0
\]
provided $\sum_{t=0}^{\infty} p_t \omega_t < \infty$.

**Proof.** Rewrite a sequential budget constraints, with $z = c - \omega$ denoting net trade, as
\[
\begin{align*}
 p_{t+1}a_{t+1} + p_t z_t &\leq p_t a_t, \\
 p_{T+1}a_{T+1} + p_T z_T &\leq p_T a_T \\
 T-1 \sum_{t=0}^{T-1} p_{t+1}a_{t+1} + p_T a_T + T \sum_{t=0}^{T} p_t z_t &\leq T \sum_{t=1}^{T} p_t a_t + p_0 a_0
\end{align*}
\]
Simplifying the first term on the left-hand-side with the first on the right-hand-side,
\[
(p_T a_T + \sum_{t=0}^{T} p_t z_t) \leq p_0 a_0
\]
\[
\limsup_T \sum_{t=0}^{T} p_t z_t \leq p_0 a_0 + \limsup_T (-p_T a_T)
\]
\[
= p_0 a_0 - \liminf_T p_T a_T
\]
By (NPG), the last term is nonnegative; hence,
\[
\limsup_T \sum_{t=0}^{T} p_t z_t \leq p_0 a_0
\]
Next, assume that $\sum_{t=0}^{\infty} p_t \omega_t < \infty$, then, the infinite sum defining the present value of lifetime consumption is finite and the result holds. \qed
Observe that (a) says that any consumption profile from an arbitrary \( t \) to a successor date \( T \) can be financed through a bond issue \( a_{t+T} \); which is exactly as saying that any consumption profile is budget feasible. Again, by Ponzi game, the household could rollover this debt without affecting future consumption, just moving forward the terminal date \( T \). Of course this may end up with an insolvent household, who will not repay her debt even in the long run (at infinity). Because, the household preferences are strictly increasing in consumption, this is exactly what will happen; as the matter of fact, the household’s problem would not have a solution, since the household would always find welfare improving to further expand consumption and debt. The NPG condition prevents this from happening and ensures that an equilibrium exists.

We now establish the reverse proposition.

**Proposition 14.** Given \( a_0 \) and \( p \), if a consumption plan \( c \) satisfies the intertemporal budget constraint and the bounded income condition,

\[
\sum_{t=0}^{\infty} p_t(c_t - \omega_t) \leq a_0, \quad \sum_{t=0}^{\infty} p_t \omega_t < \infty.
\]

then, it satisfies the sequential budget constraint and the (NPG) condition.

**Proof.** To show that the sequential budget constraint holds, let us build a portfolio \( a \) financially supporting \( c \). Truncate the intertemporal budget constraint at some date \( t \), and let

\[
a_{t+1} = \frac{1}{p_{t+1}} \left( a_0 - \sum_{s=0}^{t} p_s (c_s - \omega_s) \right), \quad \text{at all } t \text{ in } T
\]

Observe that (+) is constructed recursively, starting at \( t = 0 \), given \( z_0 \) and \( a_0 = 0 \),

\[a_1 = \frac{1}{p_1} (a_0 - p_0 z_0);\]

Then, at \( t = 2 \),

\[a_2 = \frac{p_1}{p_2} (a_1 - z_1)\]

Using the penultimate equation to eliminate \( a_1 \) in the latest, and rearranging terms,

\[\frac{p_2}{p_1} a_2 + z_1 = \frac{1}{p_1} (a_0 - p_0 z_0)\]

\[p_2 a_2 + p_0 z_0 + p_1 z_1 = a_0\]

\[a_2 = \frac{1}{p_2} (a_0 - p_0 z_0 - p_1 z_1)\]
Continue for $t = 3, 4, ..., T$, or simply use induction, to get (+). Next, we verify NPG holds. Taking the liminf of (+),

$$\liminf_{T} p T + 1 a T + 1 = a_0 + \liminf_{T} \left( - \sum_{s=0}^{T} p_s z_s \right)$$

$$= a_0 - \limsup_{T} \sum_{s=0}^{T} p_s z_s$$

$$= a_0 - \sum_{s=0}^{\infty} p_s z_s$$

$$\geq a_0 - a_0 = 0$$

where the latest uses the intertemporal budget constraint.

To finally establish the equivalence between the sequential and the intertemporal household problem, we are going to show that, thanks to the NPG condition, any individual optimal plan of a sequential economy attains a finite present value of lifetime income. Precisely,

**Proposition 15.** Let $u$ be bounded. At a solution of the household sequential problem at $p$, it must be that

$$\sum_{t \in T} p_t \omega_t < \infty$$

*Proof.* We prove it by contradiction. Let $(c, n, a)$ be a solution to the household problem at $p$ and $\omega$ be the corresponding real individual income. Suppose $\sum_{t \in T} p_t \omega_t = \infty$. Construct an alternative plan as follows. The alternative consumption plan $\tilde{c} := (1 + \epsilon)c$ with $\epsilon > 0$, and $n$ is left unchanged. Since preferences are strictly increasing in consumption, $U(\tilde{c}, n) > U(c, n)$. Let $\tilde{c}^T := (1 + \epsilon)(c_0, ..., c_T, 0, 0, ...)$ be the corresponding truncated sequence, which is $\tilde{c}$ up to $T$ and zero thereafter. Since $u$ is assumed to be bounded,

$$\limsup_{T} U(\tilde{c}^T, n) = \limsup_{T} \sum_{t=0}^{T} \beta^t u(c_t, n_t) > U(c, n)$$

Impatience ($\beta < 1$) implies that any (bounded) utility over a sufficiently distant future becomes irrelevant for the consumer. By continuity, for a sufficiently large $T^*$,

$$U(\tilde{c}^{T^*}, n) > U(c, n)$$

We are left to show that $\tilde{c}^{T^*}$ is budget feasible. To this end, construct a portfolio sequence $\tilde{a}$ as in (+) above: for $0 \leq t \leq T^*$,

$$p_{t+1} \tilde{a}_{t+1} = a_0 - \sum_{s=0}^{t} p_s (\tilde{c}_s - \omega_s)$$

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and, for any \( t > T^* \),

\[
p_{t+1} \tilde{a}_{t+1} = a_0 - \sum_{s=0}^{T^*} p_s (\tilde{c}_s - \omega_s) + \sum_{s=T^*+1}^{t} p_s \omega_s
\]

\[
= a_0 - \sum_{s=0}^{T^*} p_s \tilde{c}_s + \sum_{s=0}^{t} p_s \omega_s
\]

Taking the lim inf, and using the fact that the last term goes to \(+\infty\),

\[
\lim_{t} \inf p_{t+1} \tilde{a}_{t+1} = a_0 - \sum_{s=0}^{T^*} p_s \tilde{c}_s + \lim_{t} \inf \sum_{s=0}^{t} p_s \omega_s > 0
\]

that is \( \tilde{a} \) satisfies (NPG). We conclude that \( \tilde{a} \) satisfies (NPG) and supports a consumption sequence, \( \tilde{c}^{T^*} \), that increases the household welfare, at \((n, \omega)\) and \(p\). This contradicts the fact that the initial plan \((c, n, a)\) was individually optimal. \(\square\)

As \( \omega \) is a bounded (by \( \bar{\omega} \)) sequence, \( \sum_t p_t \omega_t \leq \bar{\omega} \sum_t p_t < \infty \), implying that present value prices are summable. Moreover, as \( p \) is nonnegative, it must be vanishing to zero over time.

We can use the results established in this subsection to conclude the following.

**Theorem 16** (I Welfare Theorem). Assume \( u \) is bounded. Every Sequential Market Equilibrium, in which the (NPG) condition holds on the household, is Pareto efficient.

8.2.5. **The II Welfare Theorem.** The II Welfare Theorem provides conditions under which a Pareto efficient allocation can be attained as a competitive equilibrium of some economy. This is established for convex economies using a separating hyperplane theorem (see, for example, Mas-Colell, Whiston and Green, chapter 14). Rather than proceed along this argument, we would only like to provide a simpler but useful argument that holds for interior efficient allocations. The argument is simple because it exploits first order conditions (above all, the Euler equation and the efficient labor supply condition). It is useful since it can be used to compute equilibrium prices easily, from allocations solving the centralized planner problem \((P)\).

Indeed, consider an interior Pareto efficient allocation \((c, n, k)\). This satisfies, \((E), (L)\) (and \((T)\)) along with the resource constraint in every date. Thus, letting \( p_0 = 1 \), the candidate prices can be defined recursively to satisfy,

\[
(p_1, p_2, ..., p_{t+1}, ...) = \left( \frac{\beta u_c(c_1, n_1)}{u_c(c_0, n_0)}, \frac{\beta u_c(c_2, n_2)}{u_c(c_1, n_1)} p_1, ..., \frac{\beta u_c(c_{t+1}, n_{t+1})}{u_c(c_t, n_t)} p_t, ... \right)
\]

and, at all \( t \) in \( \mathcal{T} \),

\[
w_t = -\frac{u_n(c_t, n_t)}{\beta u_c(c_t, n_t)} = F_n(k_t, n_t)
\]

\[
\nu_t = \frac{p_t}{p_{t+1}} - 1 + \delta = F_k(k_t, n_t)
\]
You are left to verify that allocations \((c, n, k)\) and prices satisfy the household budget constraint, either the intertemporal or the sequential, depending on the type of equilibrium considered. In case of the sequential, you will need to derive a feasible bond-portfolio, something you can do appealing to (+) above. For the firm problem, as (T) holds, you can easily show profit maximization.