The evaluation of public investments under uncertainty

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Abstract

In this paper we revise and extend the theory of the evaluation of public investments under uncertainty. Precisely, we argue that the value of the investments that the public sector would be willing to undertake is never below its market value, and that it can be higher if it provides social insurance.

Keywords: Project evaluation; Social discounting; Criteria for decision making under uncertainty; Incomplete markets

1. Introduction

There is no widely accepted theory on how public investments should be evaluated in an uncertain world. The issue at the heart of an old debate is whether the government’s position with regard to risk should be different with respect to private agents’, and if this should have implications on the evaluation of public investments. 

Hirschleifer (1964, 1966) first argued that no special role should be given to the government with respect to risk bearing if public projects are feasible for the private sector. Abstracting for motives other than allocative efficiency, if markets are competitive and complete, every project is marketable and its risk and return can be equivalently replicated either by a private project or through a portfolio of private projects. Thus, at equilibrium, there is always a security portfolio of private investments whose value coincides with that of a given public project, and this is true for all feasible public projects.

Analogous conclusions were reached by Sadmo (1972) considering economies with incomplete markets as specified in Diamond (1967). This result hinges on the assumption that production technologies are of the multiplicative class, implying that no changes in the investment policy (other than those consisting in opening or shutting down firms) may modify the market risk sharing possibilities. The same result holds, for more general technologies, in the case of partial spanning; namely, when public projects can always be replicated by a portfolio of private projects.

A second school of thought proposed more forcefully the opposite idea, that governments have a special ability to pool risk. Samuelson (1964) and Vickrey (1964) noticed that, due to the size and the large variety of their activities, public projects may offer additional risk sharing opportunities with respect to any available private one.

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1 See, for example, Arrow (1971) for an up to date introduction to this debate.
General Motors can borrow at a lower rate than American Motors because it is a puller of more independent risks. It would be absurd for G.M. to apply the same high risk-interest-discount factor to a particular venture than A.M. would apply.

Yet, the idea of ‘risk pooling’ is not completely spelled out. In a competitive economy with complete markets the first welfare theorem does not leave ground to support this thesis any further. Therefore, one should depart from the Diamond-Sadmo’s model when markets are incomplete. The literature followed different paths. Stepleton and Subrahmanyam (1978) in a mean-variance economy argued that public investments should be valued less than private ones, if they are not marketable. In fact, the impossibility of consumers to modify their share in the cost of public investments may negatively affect private risk sharing. Precisely, when all projects are discounted using the same criterion, one may observe overinvestment in public projects at equilibrium, with respect to the Pareto optimal level. Arrow and Lind (1970), instead, argued in favour of Samuelson and Vickrey’s, based on the ability of the public sector to diversify risk by dividing it through a large multitude of tax payers. Yet, one may say that their result holds not only for the case of large economies/small public investment, but for any investment project with such characteristics (private investments included).

It has been proved (Borch, 1960) that if an individual or group holds an asset which is statistically independent of other assets, and if there is one or more individuals who do not share ownership, then the existing situation is not Pareto efficient. By selling some share of asset to one of the individuals not originally possessing a share, the cost of risk-bearing can be reduced while the expected return remain unchanged.

Borch’s (1960) efficient risk-sharing rule says that every individual should hold a share of every independent asset, and this implies that his evaluation equals that of a risk neutral firm.

More recently, Grinols (1985) provides further formal analysis in support of the intuition that public projects may be valued differently from the market. However, as he recognizes, his conclusions are ambiguous: when public investments have a non-marketable component, social discounting may either be higher or lower than market discounting. Precisely, public investment should be valued more than private investment if their non-marketable component improves the allocation of risk, i.e. if they provide social insurance.

1.1. Content

In this paper we follow up on the literature mentioned by trying to provide an answer to the question of how should public investment be valued in an uncertain world. In doing so we focus on the issue of whether the government may actually have a natural ability to improve risk-sharing upon the market allocation. Similarly to competitive economies, in which pricing and discounting is a natural implication of the consumer and production theory, we make the point that the theory of public project evaluation and social discounting should also naturally stem from the theory of social choice. Since we assume that the economy has incomplete financial markets – financial markets are \( J \), less than the number of states of uncertainty \( S \) – we depart from the Pareto ordering over the set of resource-feasible allocations, to introduce a coarser definition of feasibility. A natural step in this direction is to follow Sadmo (1972) and appeal to Diamond’s (1967) notion of attainability and second best (constrained Pareto) optimality. Precisely, we assume that a public investment policy is feasible if (i) it consists of at most \( J \) projects which are achievable by the government using the technology available to the private sector, (ii) its costs and benefits can be divided among consumers using a system of personalized, non-contingent, shares. We then exploit the Pareto ordering over the resulting set of feasible allocations to construct an evaluation criterion for public investments.

Our analysis departs from Sadmo’s exactly because we assume that public investment decisions may change private risk sharing opportunities. This is achieved by the government using available technologies (i.e. without completing the markets), when public investments change the asset span.

Through the paper we try to keep our analysis at the lowest possible level of formalism. Indeed our result is simple and intuitive. Traditional corporate models of investment decisions, such as those proposed by Drèze (1974) or Grossman and Hart (1979), assume that investment projects are chosen in the interest of a given group of consumers/shareholders. In contrast, our public investment decision model ranks investments in the interest of the society as a whole, also considering that their cost-benefit distribution may change accordingly. In other words, a private investment policy is usually valued at a given distribution of costs-benefits, while a public investment policy is a simultaneous choice of investments and costs-benefits in the form of shares. This implies that the government may
Once one applies our evaluation criterion, Grinols’ (1985) disappointing conclusion is overcome. Precisely, we show that the present value of a public investment is never below the equilibrium value of any alternative, technologically feasible, investment plan (Proposition 1). We also show that it can be strictly higher than its market value if it provides additional risk-sharing opportunities (Corollary 1). These results imply that the social discount rate should never exceed private ones, and that it could be lower if the project provides social insurance.

Our work is organized as follows. In Section 2 we introduce notions of competitive economy and equilibrium. In Section 3, we provide definitions of private investment evaluation. In Section 4, we define an evaluation criterion for public investment, and an associated public investment decision rule. Finally, we establish our main result and discuss it.

2. The economy

2.1. Private agents and commodities

There are two dates indexed by 0 and 1. At date 1 there is a finite number $S$ of possible states of the world indexed by $s = 1, \ldots, S$. We also use the convention to label date 0 as $s = 0$. There is a single perishable good that can be both used for consumption and investment at date 0, and for consumption at date 1. We denote by $N = S + 1$ the number of contingent goods in the economy.

There are $H \geq 2$ consumers indexed by $h = 1, \ldots, H$. Every consumer $h$ is endowed with a (column) vector of contingent goods: $e^h := (e_0^h, e_1^h)' \in \mathbb{R}_+^N$, $e_1^h := (e_1^h, \ldots, e_S^h)^2$; where, hereafter, we use the convention of labelling with a subscript 1 the date 1, $S$—dimensional, component of a any vector in $\mathbb{R}^N$. Next, for simplicity, we assume that each individual consumption set coincides with the nonnegative orthant of the commodity space, $\mathbb{R}_+^N$, and we denote by $x := (x_0, x_1')'$ its typical element. Further, we assume that a utility function $u^h : \mathbb{R}_+^N \to \mathbb{R}$ represents the preference ordering of consumer $h$ for a particular consumption set. Finally, some rather strong, but standard, assumptions on preferences and endowments are introduced.

**Assumption 1.** For every individual $h$: (1) $u^h$ is continuous ($C^{r\geq 2}$) in $R^{S+1}_+$, strictly concave ($\forall x \in R^{S+1}_+, D^2 u^h(x)$ is negative definite), strictly increasing ($\forall x \in R^{S+1}_+, D u^h(x) \succ 0$), and boundary averse ($\forall x' \in R^{S+1}_+, u^h(x) \preceq u^h(x') \Rightarrow x \in R^{S+1}_+$); (2) endowments are strictly positive ($e^h \gg 0$).

There are $J \geq 1$ competitive firms. The technology of a typical firm $j$ is represented by its production possibility set $Y^j \subset \mathbb{R}^N$, with typical input-output vector $y^j := (-y_0^j, y_1^j)'$, with $y_0^j \geq 0$.

The properties of the production technologies are summarized in the following:

**Assumption 2.** For every firm $j$: $Y^j \subset \mathbb{R}^N$ is closed, convex, $0 \in Y^j$, the boundary of $Y^j$ is differentiable ($C^{r\geq 2}$).

Assumption 2 implies that there exists a continuous ($C^{r\geq 2}$) transformation function, $f^j : \mathbb{R}_+^N \to \mathbb{R}$, that is non-decreasing, quasi-convex, and satisfies $f^j(0) = 0$.

Finally, we let $\mathcal{Y}$ denote the Cartesian product of $Y^j$ over $J$, and $y := (y^1, \ldots, y^J)'$ its typical $NJ$—element.

2.2. Spot and security markets

There are $N ((S+1))$ spot markets in the economy. Since the only good produced is perishable, consumers’ saving can only take place through portfolio holdings (i.e. financial markets fully specify the available saving technology). Further, for simplicity, we assume that the only securities available are represented by equity contracts.

Every consumer $h$ holds an initial (column) vector of ownership shares $\gamma^h := (\gamma_1^h, \ldots, \gamma_J^h)'$ of typical element $0 \leq \gamma_j^h \leq 1$, with $\sum_j \gamma_j^h = 1$. At date 0, when financial markets open, every agent $h$ trades for a final ownership share, $\theta^h := (\theta_1^h, \ldots, \theta_J^h)'$, $0 \leq \theta_j^h \leq 1$ for all $j$. In the second period, every security $j$ pays a return $y_j^s$ to current shareholders, if state $s$ occurs. Let $q_j$ denote the market value of security $j$, $q := (q_1, \ldots, q_J)$. The market price of a

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2 We denote by $v'$ the transpose of $v$. Moreover, $v := (v^1, \ldots, v^n, \ldots, v^N) > 0$ reads $v^n \geq 0$ for all $n$; while strict vector inequality will be denoted by $\gg$.
share $\theta^j_h$ in the capital of firm $j$ is $q_j \theta^j_h$. If $h$ owns a final share $\theta^h > 0$, $h$ is also asked to finance date 0 inputs of firm $j$ proportionally, in the amount $y^j_0 \theta^h$. The matrix of returns from security holdings is,

$$W := W(q, y) = \begin{bmatrix} -y^0 - q^1 & \cdots & -y^J - q^J \\ y^1_1 & \cdots & y^J_1 \\ \vdots & \ddots & \vdots \\ y^1_S & \cdots & y^J_S \end{bmatrix} = [-y^0 - q]_{(N \times J)}.$$

Agents face the same financial structure $W$. Thus, they all have equal access opportunities to the equity market. Further, default (or bankruptcy) is not allowed.

Let $Y$ denote the $N \times J$ matrix of production plans that are traded in the economy. We say that a production plan, $y^j$, is *marketable* if $y^j$ can be expressed as a linear combination of the production plans in $Y$: there exists a nontrivial vector $\psi$ such that $y^j = Y\psi$. We define the set of marketable production plans, or *asset span*, as $(Y) := \{m : m = Y\psi, \psi \in \mathbb{R}^J\}; (Y)^\perp$ denotes the orthogonal complement of $(Y)$ in $\mathbb{R}^{S+1}$. If $y^j$ is marketable, in equilibrium, its unique evaluation can be expressed as a linear combination of the prices of the existent securities: $y^j = Y\psi$, implies that the market price of $y^j$ is $q^j = q\psi$.

2.3. Consumers

The budget set of $h$, at $(q, y, \gamma, e^h)$ is

$$\mathbb{B}(q, y, \gamma, e^h) = \{x^h \in \mathbb{R}^N : x^h - e^h = W\theta^h, \theta^h > 0, e^h := \lambda^0_0 + q y^h, e^h = (\lambda^0_0, e^h)\}.$$

A consumer optimum for $h$, at $(q, y, \gamma, e^h)$, is a pair $(x^{hs}, \theta^{hs})$ such that $x^{hs}$ maximizes $u^h(x^h)$ on $\mathbb{B}(q, y, \gamma, e^h)$, and $\theta^{hs} > 0$ satisfies $x^{hs} - e^h = W\theta^{hs}$, at $x^{hs}$. First order conditions are,

$$\lambda^h_i = D_s u^h(x^{hs}), \quad \text{for all } s$$

$$\lambda^h W = 0$$

where $\lambda^h := (\lambda^0_0, \lambda^1_h) \gg 0$, $\lambda^1_h := (\lambda^1_1, \ldots, \lambda^1_S)$, are Lagrange multipliers attached to the state-contingent budget constraints. At an individual optimum, interior by Assumption 1, the marginal rate of substitution (or state price) of consumer $h$, in state $s$, $\nabla_s u^h(x^{hs}) := D_s u^h(x^{hs})/D_{x^h} u^h(x^{hs})$, equals $\pi^{hs}_s := \lambda^{hs}_s/\lambda^0_0$, and $\pi^{hs} \in (W)^\perp$.

2.4. Firms

Let $\hat{\beta}^j := (\hat{\beta}^j_1, \hat{\beta}^j_1) \gg 0$, $\hat{\beta}^j_1 := (\hat{\beta}^j_1, \ldots, \hat{\beta}^j_1)$, be firm $j$ evaluation criterion given to firm $j$. The decision problem of firm $j$ is to choose $y^j$ such that $\hat{\beta}^j y^j$ is maximized on $\mathbb{Y}^j$. If $y^j \gg 0$ solves $j$’s maximization, first order conditions imply,

$$\hat{\beta}^j_1 = D_s f^j(y^{j*}), \quad \text{for all } s.$$

We also use $\beta^j_s := \hat{\beta}^j_s/\hat{\beta}^j_0$, and $\nabla_s f(y) := D_s f^j/D_{x^j} f$.

In principle, we would (at least) like to require that, at equilibrium, $\hat{\beta}^j$ is *consistent* with the market value of the firm, $q^j$; where by consistent we mean that, at an interior firm’s optimum, $y^j, q^j = \beta^j y^j$ (see De Marzo (1988)). For example, we may assume that firm $j$ acts in the ‘best interest of shareholders’ by letting $\beta^j$ be a (linear) function of shareholders state prices, $\pi^h$. Then, any consistent $\beta^j$ leads to a well defined firm’s problem.

2.5. Competitive equilibrium

We provide the following, standard, definition of equilibrium.

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3 A further necessary condition which is more related to the fact that individual demands are well behaved, in that, in every state $s$, (at least) one consumer, $h(s)$, has nonsatiation preferences. This ensures that in equilibrium $\lambda^{hs}(s) > 0$, and thus that all firms are valued, preventing the security matrix to dropping rank.
Definition 1 (β-Competitive Equilibrium). In an economy $E (u, f, \eta, e, \gamma)$ a β-competitive equilibrium is a pair $(x, \theta, y, q)$ such that

(i) $\forall h, (x^h, \theta^h)$ solves the consumer problem of $h$, at $(q, y, \gamma^h, e^h)$;

(ii) $\forall j, y^j$ solves the firm problem of $j$, at $\beta^j$, and $\beta^j \in \langle W(q, y) \rangle$;

(iii) $\forall j, \sum_h \theta^j_h = 1$.

Observe that if (i)–(iii) are satisfied, spot markets clear. Finally, we say that an equilibrium is interior if its corresponding allocation has nonzero elements, $(x, \theta, y_0, y_1) \gg 0$.

3. Private evaluation of investments

In a competitive economy the evaluation criterion of investments translates in the role: invest in input $y^j_0$ if $q^j, h > q^j$, and continue to invest until the latter holds with equality. In fact, at an interior equilibrium, no-arbitrage is achieved when $q^{j, h} = \sum_s \pi^h_s y^j_s = \sum_s \beta^j_s y^j_s = q^j$.

If financial markets are complete, $Y_1$ has rank $S$ and consumers perfectly agree on projects' evaluations state-by-state: $\nabla u^h = \nabla y^h, \forall s, h$, i.e. as in a standard Arrow–Debreu economy consumers', individual optimality implies that normalized gradients are identical. In fact, let $(Y_1)$ denote the subspace generated by the columns of $Y_1$, and $(Y_1) \perp$ its orthogonal in $\mathbb{R}^S$. With complete markets, $(Y_1)$ and $(Y_1) \perp$ are of dimension $S$ and $0$ respectively. Because, at an interior optimum, $\pi^h = \nabla u^h$, and $(\pi^h - \pi^1) Y_1 = 0$, it must be that $\pi^h = \pi^1$, for all $h$.

Therefore, if markets are complete, and firms act in the best interest of shareholders, it must be that, for given $\pi (=\pi^1)$, a production plan $y^j$ is chosen such that $y^j$ maximizes $\pi y^j$ on $Y^j$, for all $j$. By introducing this objective function, it is straightforward to show that the corresponding equilibrium allocation is Pareto optimal.

This objective function is also appropriate in the case of partial spanning. Partial spanning occurs when financial markets are incomplete, but firms are constrained to propose projects which exclusively lie in the span of the marketed securities (i.e. for all $j$, $y^j_1$ is contained in $(Y_1)$, and dim $(Y_1) = J \leq S$). This condition implies that the return of every new project is marketable.

When markets are incomplete, $J < S$, consumers do not typically agree on the evaluation of future income profiles. True $(\pi^1_1 - \pi^1) Y_1 = 0$ must hold at equilibrium, but if not $(Y_1)$ has dimension $J < S$, $(Y_1) \perp$ has a strictly positive dimension, $S - J$. The latter implies that $\pi^h \neq \pi^1$ may (and typically does) hold at equilibrium. Since consumers are also shareholders, the latter poses some problems for the definition of the objective function of the firm. Precisely, if the firm chooses its production plans in the best interest of shareholders, none of its feasible choices will, typically, achieve shareholders’ unanimous approval. This point is addressed in the following subsection.

3.1. Corporate evaluation of investments

Suppose that a firm has a partnership structure: shares are divided among a group of consumers, shareholders, who own and control the firm. Two consistent criteria, $\beta$, have been extensively used in the literature. One was proposed by Drèze, and the second by Grossman and Hart. In Drèze (1974) each firm takes investment decisions in the best interest of final shareholders, by discounting future profits at $\beta^j = \sum_h \theta^h \pi^h$; i.e. the value of a production plan $y^j$ is $\beta^j y^j = \sum_h \theta^h q^{h, j}$. In Grossman and Hart (1979), firms act in the best interest of initial (or date 0) shareholders, and $\beta^j = \sum_h \gamma^h \pi^h$; i.e. the value of a production plan $y^j$ is $\beta^j y^j = \sum_h \gamma^h q^{h, j}$.

These two decision criteria mainly differ in the ‘timing’ at which the evaluation of production plans is made. In Grossman and Hart projects are chosen when asset markets are still open. Thus shareholders may always ‘vote’ against an investment plan by selling their shares in the firm. At an interior equilibrium, the no-arbitrage condition ensures that the market price they are paid matches their private evaluations. On the contrary, Drèze’s criterion is based on the idea that production plans are decided after the security market has met and the ownership structures have been decided. Thus, the interests of current (or final) shareholders may only be protected by letting ownership implying production control.

Notice that the fact that $\beta^j$ is given to the firm captures its competitive behaviour; strategic firms would take into account the effect of their choice on state prices $\beta^j$. In Grossman and Hart this assumption is paired with competitive price perception; that is, each consumer $h$ holding a share of firm $j$ anticipates that a change of the production plan
\(\Delta y^j\) will affect its market price \(q^j\) only through direct capital gains and losses. No other effects accruing from \(\Delta y^j\) would be taken into account.

A few formal definitions may help the reader.

**Definition 2** *(Firm *j* Constrained Efficient Decision)*. Let \((x^*, \theta^*, y^*)\) be individually optimal actions for consumers and firms at a given economy. The decision \(y^j*\) is firm *j* constrained efficient (at \((x^*, \theta^*)\)) if there does not exist a \(y^j \in \mathcal{Y}^j\) and transfers, \(\tau^h\), among shareholders of *j*, such that the associated change in consumption, \(\Delta x^h\), satisfies \(u^h(x^* + \Delta x^h) \geq u^h(x^*)\) for all \(h\), with strict inequality for some \(h\) who is shareholder of *j*, and \(\sum_{h \in \mathcal{H}_j} \tau^h \geq 0.4^\)

Observe that here the term ‘efficiency’ is used referring to the owners of a specific firm, *j*; elsewhere this criterion has been addressed as ‘shareholders constrained efficiency’. Moreover, our definition is not complete in so far as it does not make precise which variables may adjust after a production change, \(\Delta y^j = (y^j - y^*)\), occurs. This adjustment, ultimately, depends on the institutional structure and on the possibility of consumers to modify their saving decisions on the security markets when production decisions are taken. Thus, while in Grossman–Hart’s both consumption and portfolio adjustments are considered, in Dréze’s production decisions are taken after security markets close, and thus do not allow for regrading in shares.

**Definition 3** *(Firm *j* Locally Constrained Efficient Criterion)*. \(\beta^j\) is a firm *j* constrained efficient criterion, at \(y^*\), if it supports a firm *j* constrained efficient decision, \(y^*\), as a profit maximum for *j*.

More formally, \(\beta^j(y^*)\) is firm *j* (locally) constrained efficient criterion if \(y^*\) maximizes \(\beta^j(y^j)\) on \(\mathcal{Y}^j\), and \(y^*\) is also firm *j* constrained efficient in the sense of Definition 2.

Both Dréze’s and Grossman–Hart’s criteria can be shown to be firm *j* (locally) constrained efficient criteria. For Grossman–Hart’s one needs to assume competitive price perception; for both criteria it is important to remember the implicit assumptions made on the timing of production and trade.

**Fact 1.** *Let Assumptions 1 and 2 hold, production decisions are taken after security markets close, and date-zero side-payments among shareholders be allowed. Then, \(\beta^j\) is firm *j* locally constrained efficient criterion if and only if \(\beta^j = \sum_h \theta^h \pi^h\).*

(See Appendix for a formal proof.)

The equivalent of the last fact can be also established for the Grossman–Hart criterion.

**Fact 2.** *Let Assumptions 1 and 2 hold, production decisions are taken when security markets are still open, consumers have competitive price perceptions, and date-zero side-payments among shareholders be allowed. Then, \(\beta^j\) is firm *j* locally constrained efficient criterion if and only if \(\beta^j = \sum_h y^h \pi^h\).*

(See Appendix for a formal proof.)

Finally, we define a Dréze equilibrium (respectively, a Grossman–Hart equilibrium) a \(\beta\)-equilibrium with \(\beta\) taken to be the Dréze (resp. Grossman–Hart) criterion.

**Remark 1** *(Grossman–Hart Equilibrium)*. When \(\beta\) is taken to be Grossman and Hart’s firm criterion, and competitive price perceptions are assumed, our notion of \(\beta\)-competitive equilibrium must be adjusted correspondingly. Price conjectures must be formally taken into account.

**Remark 2** *(Complete Markets)*. A particular case is one in which there are as many assets as states of uncertainty \((J = S)\). When asset markets are complete, at equilibrium, both constrained efficient criteria reduce to \(\beta^j = \pi^h = \pi\) for all \(h\), and *j*. This follows trivially from the observation that, the allocation of a GEI equilibrium with \(J = S\) assets, typically, coincides with an Arrow–Debreu equilibrium allocation.

\(^4\)The assumption that \(\sum_{h \in \mathcal{H}_j} \tau^h \geq 0\) should be understood to mean that the aggregate contribution of those shareholders who benefit from a change in the production plan are enough to subsidize those who are damaged.
4. Public evaluation of investments

As for private agents, to determine an evaluation criterion for public investment projects one should specify the setting in which a ‘central planner’, or government, is called upon to make decisions; hence, which are the planner’s objective, controls and constraints?

Although this leaves room for different assumptions, a natural starting point is to assume that (i) a government ranks investment projects according to the Pareto criterion, (ii) an investment project \( y \) is feasible if it belongs to \( \mathcal{Y} \), (iii) the distribution of investments cost-benefits is also decided by the government at the time investments are undertaken.

Observe that an important implication of requirement (ii) is that – as for the private sector – the government undertakes, \( \mathcal{Y} \) is in \( \mathcal{Y} \) and thus (in \( \mathcal{Y} \)) and the central government assigns equal welfare weights, of 1, and that an economy \( e \) ranks investment projects according to the Pareto criterion, (ii) an investment project \( y \) is feasible if it belongs to \( \mathcal{Y} \), (iii) the distribution of investments cost-benefits is also decided by the government at the time investments are undertaken.

To rank or evaluate alternative, feasible, public investment projects we use the Pareto ordering over the set of those consumption allocations which can be achieved by implementing all feasible investment policies. Thus, we first define the set of attainable consumption allocations at an investment project \( y \) of an economy \( e \) as

\[
\mathcal{A}(y, e) := \left\{ x : \forall j, \sum_h \theta_j^h - 1 = 0 \right\}.
\]

Next, let \( \tilde{\mathcal{Y}} \) be the subset of \( \mathcal{Y} \) whose elements \( y \) form a full rank matrix \( Y \). Clearly, under our maintained assumptions, for every \( y \) in \( \tilde{\mathcal{Y}} \), \( \mathcal{A}(y, e) \) is nonempty, compact and convex. For given welfare weights \( \delta \in \mathbb{R}^{H-1} \), a Pareto optimal allocation \( x \) at \( (y, e) \) is a solution to the maximum problem:

\[
\max_{x \in \mathcal{A}(y, e)} \sum_h \delta_h u^h(x^h).
\]

A solution to this problem, \( x(y, e) \), exists and is unique for all \( y \) in \( \tilde{\mathcal{Y}} \) and all \( e \). Moreover, by the implicit function theorem, \( x(y, e) \) and the cost-benefit share \( \theta(y, e) \) are continuous. Indeed if \( Y \) has rank \( J \), there exists an invertible submatrix \( \tilde{Y} \in \mathbb{R}^{J \times J} \), such that \( \theta(y, e) := \tilde{Y}^{-1}(x^h(y, e) - \bar{v}^h) \) for all \( h \), where \( x^h(y, e) \) and \( \bar{v}^h \) are \( J \)—sub-vectors of \( x^h(y, e) \) and \( \bar{v}^h \), conforming to \( \tilde{Y} \).

We are now ready to apply the Pareto criterion to rank alternative investment projects (and policies). To this end, observe that the overall set of attainable consumption allocations is the union of such sets on \( \mathcal{Y} \): \( \mathcal{A}(e) := \bigcup_{y \in \tilde{\mathcal{Y}}} \mathcal{A}(y, e) \). Then, for any pair of technologically feasible investments, \( (y^*, y^0) \) in \( \tilde{\mathcal{Y}} \), we say that \( y^* \) should be valued more than \( y^0 \), if there exists a share distribution \( \theta^* \) (in \( \theta(y^*, e) \)) and a consumption allocation \( x^* \) (in \( x(y^*, e) \), and thus) in \( \mathcal{A}(y^*, e) \) that Pareto dominates any consumption allocation in \( \mathcal{A}(y^0, e) \). If, for simplicity, we assume that the central government assigns equal welfare weights, of 1, and that an economy \( e \) is fixed\(^6\) (so that we can drop \( e \)

\(^5\) Fix \((y, e)\). If \( x \) is attainable in the sense of: Diamond (1967) at \((y, e)\),

\[
\sum_h (x^h_0 - e^h_0) + \sum_j y^j_0 \leq 0
\]

\[
\forall h, x^h_1 - e^h_1 - Y^1 \theta^h = 0, \theta^h \in [0, 1]^J
\]

\[
\forall j, \sum_h \theta_j^h - 1 = 0.
\]

then it does also satisfy \( x^h_0 - e^h_0 + \sum_j y^j_0 \theta_j^h - \tau^h \leq 0 \) for some system of taxes and side-payments, \( (\tau^1, \ldots, \tau^H) \), \( \sum_h \tau^h = 0 \). The latest is equivalent to say that \( x \) is in \( \mathcal{A}(y, e, \tau) \), where \( \mathcal{A}(y, e, \tau) \) is defined as \( \mathcal{A}(y, e) \) with the first \( H \) conditions substituted by: \( x^h_0 - e^h_0 + \sum_j y^j_0 \theta_j^h - \tau^h \leq 0 \) for all \( h \). It is immediately seen that if \( x \in \mathcal{A}(y, e, \tau) \) then it is attainable in the sense of Diamond.

\(^6\) Precisely, we should say that we let \( e \) to be a regular economy, i.e. an economy such that an equilibrium exists, it is regular, and \( y(e) \) is in \( \tilde{\mathcal{Y}} \). See Geanakoplos et al. (1990).
from our notation), our Pareto ranking leads to the following evaluation criterion. A necessary condition for a public investment project \( y^* \) to be valued more than \( y^0 \) is that \( x(y^*) \) is in \( \mathcal{A}(y^0) \), and

\[
\sum_h u^h(x^h(y^*)) > \sum_h u^h(x^h(y^0)).
\]

Using strict concavity and differentiability of individual utilities, this is true if and only if

\[
\sum_h \nabla u^h(x^h(y^0)) \cdot (x^h(y^*) - x^h(y^0)) > 0.
\]

Then, denoting \( \nabla y^0 := (1, \nabla_1 u^h(x^h(y^0))) \), we find that a public investment \( y^* \) is valued more than \( y^0 \) if and only if:

\[
\sum_j \left[ \nabla y^0 \cdot y^* + \sum_h \theta^h(y^*)(\nabla y^0_1 - \nabla y^0_1) \cdot y_1^j \right] - \sum_j \left[ \nabla y^0 \cdot y^j + \sum_h \theta^h(y^0)(\nabla y^0_1 - \nabla y^0_1) \cdot y_1^j \right] > 0. \tag{2}
\]

Using \( \sum_h \theta^h = \sum_h \theta^h = 1 \),

\[
\sum_j \sum_h \theta^h(y^*)(\nabla y^0_1 - \nabla y^0_1) \cdot y_1^j > 0,
\]

Notice that, while all the variables denoted with a superscript 0 are invariant with respect to the choice of \( y^* \), the cost-benefit share, \( \theta(y^*) \), is clearly not so. This is to say that, given an initial state of the economy, defined by a pair \( (y^0, \theta^0) \), a planner that is called upon to evaluate an alternative public investment \( y^* \) should also consider that its costs and benefits may be divided among agents according to a share \( \theta^* \) in \( \theta(y^*) \), possibly different from \( \theta^0 \).

This latter discussion leads to define a relatively simple public investment rule. For an economy with an initial investment policy and associated cost-benefit share, \( (y^0, \theta^0) \), a public investment policy is a pair \( (y^*, \theta^*) \) such that \( y^* \) solves,

\[
\text{Max}_{y^j \in \mathcal{Y}^j} \sum_h \theta^h(y^j) \nabla y^0 \cdot y^j, \quad \text{for all } j \tag{3}
\]

and \( \theta^* \) is in \( \theta(y^*) \). Indeed, for \( y^* \) sufficiently close to \( y^0 \), this condition is equivalent to maximize the aggregate welfare gain of moving away from \( y^0 \), evaluated at the time of investment (i.e. at date 0). An investment \( y^* \) decided according to this rule allows to achieve a Pareto optimum on \( \mathcal{A} \) (i.e. a second best, or a ‘constrained Pareto optimum’ in the sense of Diamond (1967)) provided that the central government can make transfers (or side-payments) in date 0.

The reader may have noticed that the value maximization in (3) is similar to the firm decision criterion in Drèze (1974), in that it weights production plans using portfolios and individual state-prices (or marginal rates of substitutions). However, our criterion is substantially different from Drèze’s for two important reasons. First, it values investments in the perspective of every agent in the economy, as supposed as in that of current shareholders. Second, as pointed out above, it weights shareholders with portfolios which are themselves functions of the chosen production plans. The latter implies that, unlike for the individual firm in Drèze’s, the planner’s value maximization is a nonlinear function of investment projects.

### 4.1. Comparing public and private evaluations

Having defined the meaning of public investment policies, we now turn to compare public and private investment decisions. Our comparison is simple and hinges upon two considerations. First, it is immediately seen that every individually optimal allocation is attainable by the planner. Therefore, our central government can always replicate the investment decisions and cost-benefit allocation achievable in that economy at a competitive equilibrium. Second, let \( (x^0, \theta^0, y^0, q^0) \) be a competitive equilibrium with associated marginal rates of substitution \((..., \nabla h, ..., \)\), and \( \theta^h \) a value of \( \theta^h(y^*) \); (3) implies that

\[
\sum_h \theta^h \nabla y^0 \cdot y^* \geq \sum_h \theta^h \nabla y^0 \cdot y^j \tag{4}
\]
for all investment plans \( y^j \in \mathcal{Y}^j \), all cost-benefit shares \( \theta_j \), and all technologies \( j \). In words, a public investment project should never be assigned a lower present value than any private one. Indeed, let \( y^* \) be achieved at an equilibrium, \( (x, \theta, y^*, q) \), with associated marginal rates of substitution \( (\ldots, \nabla^h, \ldots) \). (3) implies that \( \sum_h \theta_h^h \nabla^h \cdot y^{j^*} \geq 0 \), for all \( j \), and thus,

\[
\sum_h \left( \theta_h^h \nabla_1^h - \theta_j^h \nabla_1^h \right) \cdot y^{j^*} \geq 0, \quad \text{all } j.
\]  

(5)

In words, even if private agents effectively choose to implement an investment plan \( y^* \), that would be the right candidate for a public investment policy, the present value of \( y^* \), in the eyes of private agents, may be lower than the one computed by the government. This implies that in order to implement \( y^* \) private agents may be willing to give up fewer date 0 resources than the government.

This whole discussion can be summarized in the following.

**Proposition 1.** Suppose that the government has access to the same technologies owned by the private sector, and that it evaluates investments according to our public investment rule. Then, the social value of any public investment is never below its market value.

Special cases in which the value of public projects coincides with private are the following:

1. economies in which agents agree on the evaluations of projects: \( \nabla_1^h = \nabla_1^1 \), all \( h \);
2. technologies are multilinear;
3. every public project is restricted to be a linear combination of existing projects (i.e. partial spanning).

In (1) market incompleteness is effectively irrelevant, and agents achieve Pareto optimality. This is a nongeneric situation; namely, it does not survive to perturbations of endowments. (2) implies that public investments do not affect risk sharing and that the market equilibrium allocation is constrained Pareto optimal (see Diamond (1967)). (3) is also very peculiar since – beside the case in which technologies are multilinear – there is no plausible reason why public investments should not in general provide additional risk sharing possibilities with respect to those already offered on the market. Because of market incompleteness, public investments do typically affect risk sharing, and this is a crucial fact in determining their values. To clarify this point, let us decompose projects and marginal rates of substitutions on the asset span and its orthogonal subspace. Precisely, for every \( \nabla^h \) and \( y^j \) we have, \( \nabla^h = \hat{\nabla} + \nabla^h \perp \), and \( y = \hat{y}^j + y^j \perp \), where the first component, with the “cup” sign, belongs to the column span of \( W \), and the second (\( \perp \)) to its orthogonal space. Then, \( \nabla_1^h y_1 = \hat{\nabla}_1^1 \hat{y}^j + \nabla_1^h y^j \perp \), implying that (5) holds if and only if \( \sum_h \left( \theta_h^h - \theta_j^h \right) \nabla_1^h \cdot y^j \perp \geq 0 \), all \( j \).

In words,

**Corollary 1.** For a public evaluation of a project to be strictly higher than its private evaluation it is necessary that the project provides social insurance.

We conclude our analysis with the following three remarks:

**Remark 3 (Social Discounting).** Proposition 1 implies that the implicit discount rate derived for a public investment project should never exceed the corporate discount rate. Moreover, by Corollary 1, it may be strictly lower if the public project provides social insurance.

**Remark 4 (Cost-benefit).** Although we assume that the central government directly controls the cost-benefits distribution of public investments, this is not essential to obtain our result. Indeed, we could have instead designed an indirect distribution scheme based on taxes and subsidies. To grasp the idea of our argument assume that rather than directly choosing cost-benefit shares, \( \theta \), a government policy consists in choosing an investment policy and a system of income redistributions, \( (y, m) \). Further, restrict the income redistribution \( m \) to use the available asset structure\(^7\): \( m := (m^1, \ldots, m^H) \in \mathbb{R}^{SH} \), \( \sum_h m^h = 0 \), is feasible if there exists a \( \psi := (\psi^1, \ldots, \psi^H) \), \( \sum_h \psi^h = 1 \), such that

\(^7\) We assume this to avoid the more trivial case of a government that has greater potential to sharing risk than the private sector.
Then – restricting attention to matrices $Y_1$ of full rank $J$ – one can easily recover $\psi^h$ from $m^h$ and $Y_1$. $\psi$ can be used instead of $\theta$ to evaluate investment projects according to (3), and the whole analysis goes through.

Remark 5 (Implementation). Our analysis and results on the evaluation of public investments are subject to the same criticisms raised for Drèze (1974) and Grossman and Hart (1979). First of all, computing the value of investments based on these criteria is compelling, because it relies on the observability of individual preferences, or state prices. This surely is an important problem for their actual implementation. Indeed, recoverability of individual characteristics is particularly problematic when markets are incomplete; for example, preferences are recoverable if one can know every agent’s entire excess demand function, including out of equilibrium, Geanakoplos and Polemarchakis (1990); or know the equilibrium prices that would prevail if the endowments were perturbed in an arbitrary but small direction, Kübler et al. (2002). A milder criticism concerns the requirement that the agent deciding for an investment plan be allowed to make individual side-payments. The idea behind Drèze–Grossman–Hart is that if the value of the firm is not maximized, given their firm objective, there is room for a ‘takeover’ (or an overturning of the management board) scheme. In fact, such a plan can be ‘blocked’ by a ‘new manager’ proposing a different plan that elicits money from those shareholders who would be better-off and use it to subsidize those who would be worse-off. This ‘complicated’ scheme requires shareholders unanimity. Yet, this requirement can be weakened, by adopting mechanisms relying on majority voting. Recently, Crès and Tvede (2005) derived conditions under which a super-majority voting mechanism can implement a Drèze or a Grossman–Hart equilibrium with side-payments. Clearly, one can think of analogous voting mechanisms for public decision making. For example, in local political elections, it is reasonable to think of agents (citizens) who actually vote based on their personal evaluations of competing platforms of benefits (public services and activities) and costs (their activation or investment costs somehow distributed).

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Appendix

The following proofs are added for completeness, since they revisit the arguments of Proposition 2 in Geanakoplos et al. (1990).

Proof of Fact 1 (Necessity). Let $(x^*, \theta^*, y^*)$ be individually optimal actions for consumers and firms at a given economy, and $\beta^{j*} := \sum_h \theta^h \pi^h \psi^h$, $\pi^h := \nabla u^h(x^h)$. We want to show that $y^{j*}$ being a profit maximum for $j$ at $\beta^{j*}$, implies that $y^{j*}$ is a constrained efficient decision for $j$. Proceeding by contradiction, let $y^{j*}$ not be a constrained efficient decision for $j$. There exists a production vector $y^{j}$ in $\mathcal{Y}^j$, and date zero side-payments $(\tau^h)_{h \in \mathcal{H}_j}$, $\sum_{h \in \mathcal{H}_j} \tau^h = 0$, such that $\Delta x^h = (y^{j} - y^{j*})\theta^h + \tau^h t_0$, $t_0 := (1, 0, \ldots, 0)^T \in \mathbb{R}^{\mathcal{H}_j}$, and $u^h(x^h + \Delta x^h) \geq u^h(x^h)$ for all $h$, with strict inequality for some $h$, say $h'$. Then there are state prices $\pi^h = \pi^{h*} \psi^h$ such that $\pi^{h*} \Delta x^h > 0$ for all $h$, with strict inequality for $h'$. Therefore, $\sum_h \pi^{h*}(y^{j} - y^{j*})\theta^h = \beta^{j*}(y^{j} - y^{j*}) > 0$ which implies that $y^{j*}$ is not profit maximizing at $\beta^{j*}$; a contradiction.

(Sufficiency) Proceeding by contradiction, assume that there exists a $y^{i}$ in $\mathcal{Y}^i$ such that $\beta^{i*}y^{i} > \beta^{i*}y^*$, i.e. $\sum_h \theta^h \pi^h (y^{i} - y^{i*}) > 0$. Let side-payments be defined as, $\tau^h = \delta^h \pi^h (y^{i} - y^{i*}) - \epsilon$, $\epsilon > 0$, for all $h$. For $\epsilon$ small enough, $\sum_h \tau^h = 0$, and $\delta^h \pi^h (y^{i} - y^{i*}) - \tau^h t_0 = \epsilon > 0$ implies that $\pi^h \Delta x^h > 0$, $\Delta x^h := (\theta^h (y^{i} - y^{i*}) - \tau^h t_0)$, for all $h$. Finally, for all $h \in \mathcal{H}_j$, $\nabla u^h(x^h) \Delta x^h > 0$ and monotonicity of $u^h$ implies $\psi^h = (\theta^h (y^{i} - y^{i*}) - \tau^h t_0)$.

8 The argument follows the one used to define $\theta(y, \epsilon)$. Suppose that $Y$ has rank $J$, then there exists a subset of $J$ rows and an invertible submatrix $\mathcal{T} \in \mathbb{R}^{J \times J}$, composed by such rows. Then, for any state-contingent system of taxes and subsidies $m \in \mathbb{R}^{MH}$, with the restriction that $m^h = Y_1 \psi^h$ for some $\psi^h$ and all $h$, we can express the latter as: $\psi^h := \sum_{j=1}^J \mathcal{T}^{-1} m^j$ for all $h$. This, in turn, implies that $m^1_1 = Y_1 \mathcal{T}^{-1} m^h$; that paired with budget balance ($\sum_h m^h = 0$) yields a number of instruments that is still $J (H - 1)$.

9 Quotes were from Grossman and Hart (1979), p. 301.

10 Under our assumption, $\pi^h = \psi^h (1, \psi^h)$ is unique. Assuming that $u^h$ are continuous, quasi-concavity, strictly monotone, $\pi^h$s still exists. This is shown applying standard separation arguments in (Geanakoplos et al., 1990), Proposition 1.
that there exists a $\mu \in (0, 1]$ such that $u^h(x^* + \mu \Delta x^h) > u^h(x^*)$. Since $Y^j$ is convex, there exists a $y^{j'} = \mu(y^j - y^{j\ast})$ such that $y^{j'} \in Y^j$. Let $\tau^h = \mu \tau^h_0$, $\mu \Delta x^h = \theta^h j y^{j'} - \tau^h_0$, and $u^h(x^* + \theta^h j y^{j'} - \tau^h_0) > u^h(x^*)$ for all $h$ in $\mathcal{H}_j$, and $\sum_{h \in \mathcal{H}_j} \tau^h \geq 0$. Thus, $y^{j'}$ is strictly preferred to $y^{j\ast}$ and it is attainable using nonnegative aggregate side-payments (i.e. $y^{j\ast}$ is not locally constrained efficient for $j$).

Proof of Fact 2. The same arguments used to prove Fact 1 apply here, after having observed that now we assume that asset retrade is possible, after a production change occurs. That is: $\Delta x^h_0 = -(q^j + y^j_0 + \Delta q^j_0 - \Delta q^j_0 + \Delta y^j_0) - \Delta q^j_0 + \Delta y^j_0 + \Delta y^j_0$, with competitive price perception implying that adjustments of price conjectures satisfy $\Delta q^j = \pi^h \Delta y^j$. 

References