HYDROLOGICAL PROCESSES

Analysis of the nonlinear storage–discharge relation for hillslopes through 2D numerical modelling

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Abstract:

Storage–discharge curves are widely used in several hydrological applications concerning flow and solute transport in small catchments. This article analyzes the relation \( Q(S) \) (where \( Q \) is the discharge and \( S \) is the saturated storage in the hillslope), as a function of some simple structural parameters. The relation \( Q(S) \) is evaluated through two-dimensional numerical simulations and makes use of dimensionless quantities. The method lies in between simple analytical approaches, like those based on the Boussinesq formulation, and more complex distributed models. After the numerical solution of the dimensionless Richards equation, simple analytical relations for \( Q(S) \) are determined in dimensionless form, as a function of a few relevant physical parameters. It was found that the storage–discharge curve can be well approximated by a power law function \( Q(LK_0) = aS^{b} \left( L^2(\phi - \theta_e)\right)^{\phi} \), where \( L \) is the length of the hillslope, \( K_0 \), the saturated conductivity, \( \phi - \theta_e \), the effective porosity, and \( a, b \) two coefficients which mainly depend on the slope. The results confirm the validity of the widely used power law assumption for \( Q(S) \). Similar relations can be obtained by performing a standard recession curve analysis. Although simplified, the results obtained in the present work may serve as a preliminary tool for assessing the storage–discharge relation in hillslopes. Copyright © 2012 John Wiley & Sons, Ltd.

KEY WORDS hillslope; nonlinear reservoir; flow recession; storage–discharge relation

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INTRODUCTION

The storage–discharge relation is a very important component of catchment hydrology and it is widely used for several engineering applications, such as estimating design floods (Rahman and Goonetilleke, 2001), forecasting of low flows for water resource management (Vogel and Kroll, 1992), estimating groundwater potential of basins (Wittenberg and Sivapalan, 1999) and rainfall runoff models (Srivongsitanon et al., 1998). The matter has been investigated in the framework of base flow recession, hydrograph separation and other related areas of hillslope hydrology (Brooks et al., 2004; Weiler and McDonnell, 2004; Fiori and Russo, 2007; Graham and McDonnell, 2010; McGuire and McDonnell, 2010). Based on different principles and approaches, the recession of subsurface flow has been studied by Brutsaert and Nieber (1977), Wittenberg (1994, 1999), Fenicia et al. (2006), Aksoy and Wittenberg (2011), Wang (2011) and Moore (1997). Tallaksen (1995) has discussed various methods and approaches widely used to determine the storage–discharge relationship by the recession curve analysis.

One of the widely used storage–discharge relations is the linear reservoir model, originally defined by Mailet (1905), which implies that the aquifer behaves like a single reservoir with storage \( S \), linearly proportional to outflow \( Q \), namely \( Q = aS \). In this case, the plot of \( \log Q \) against \( S \) yields a straight line (Barnes, 1939). Moreover, Fenicia et al. (2006) have also derived an actual \( S–Q \) relation in which the percolated water from the unsaturated reservoir as well as the preferential recharge as inflow to the saturated reservoir are taken into account, for which the \( S–Q \) relation was treated as a linear and second-order polynomial. However, linear reservoirs \( (Q = aS) \) can very well describe the groundwater behaviour for most of the catchments they have studied. In most actual cases, however, semilogarithmic plots of flow recessions are still concave (Aksoy and Wittenberg, 2011), indicating nonlinear storage–discharge relationships.

Recent studies agreed that the outflow of a lumped storage model can be characterised by a general power law function, \( Q = aS^b \), where \( a \) and \( b \) are constants. The constant \( b \) varies between 0 and 2 or higher for some cases (Chapman, 1997; Wittenberg, 1999; Wittenberg and Sivapalan, 1999; Harman and Sivapalan, 2009) and it has also been proved by physical experiments (Wittenberg, 1994; Chapman, 1999). The power law formulation is only occasionally chosen in recession analysis (Wittenberg, 1994), and the recession process is commonly formulated in terms of the reservoir inflow and outflow, which can be calculated using the continuity equation.

On the other hand, Brutsaert and Nieber (1977) have proposed to determine the outflow rate from a low-flow hydrograph derived from direct measurements of \( Q(t) \). They have developed their models based on the nonlinear...
Dupuit–Boussinesq aquifer model and determined the parameters for the lumped storage model. This low-flow analysis has been extensively applied by various researchers. The storage model they have used has the form of a power function, \( dQ/dr = f(Q) = -aQ^b \), where \( a \) and \( b \) are constants and can also be obtained from the Dupuit–Boussinesq aquifer model by assuming some geometrical similarity of the catchment. From the solution for the outflow rate, they have estimated \( b_1 = 3 \) and \( b_2 = 3/2 \) for short time and long time solutions, respectively.

Recently, Wang (2011) used the method proposed by Brutsaert and Nieber (1977) at the Panola Mountain Research Watershed, Atlanta, GA. The work has elaborated on the effect of groundwater leakage and return flow on the recession curves. In his work, he studied the recessions of three nested hillslopes and watersheds and estimated the recession slope curves for different values of \( a \) and \( b \) on the basis of the observed data for the three watersheds at Panola Mountain.

Parallel to the analytical approximation for computing hillslope subsurface flow, numerical models are increasingly used for computing the recession flow and base flow analysis, and also for comparison with the analytical approximation (Szilagyi et al., 1998; Lee, 2007; Rocha et al., 2007). For example, Lee (2007) develops a recession model that can provide the theoretical basis of subsurface modelling using dimensionless Richards equation, treating both saturated and unsaturated flow domains.

The use of the storage–discharge relationship, together with the continuity equation, resembles other similar approaches in hydrological modelling, such as the kinematic model for flood propagation. In essence, the dynamic equation is simplified by a quasi-steady-state relation between storage and discharge. The same approach was used by Kirchner (2009) and Botter et al. (2009, 2010).

This work aims to determine the nonlinear storage–discharge relationship by numerical computations carried out using a two-dimensional (2D) model. The proposed approach tries to develop simple solutions, with a minimal set of physical parameters, which are as parsimonious and simple as the analytical approaches available in the literature, but relaxing some of the assumptions, such as the boundary conditions and the hydrostatic Dupuit hypothesis. The relation \( Q(S) \) is expressed as a function of some simple and dimensionless structural parameters. Although the method embeds some simplifications, it may constitute a preliminary tool for assessing the nonlinear storage–discharge relation to be used in hydrological flow models.

**MATHEMATICAL FRAMEWORK AND NUMERICAL SOLUTION**

We simplify the hillslope configuration as a sloped parallelogram shape domain. The conceptual model of the hillslope is shown in Figure 1. The domain has inflow boundary at the upper face (net precipitation, i.e. subtracted from evapotranspiration), no flow at the left side and lower face, and a mixed boundary condition at the right face of the domain, namely the seepage face. In this study, we concentrate on flows driven by groundwater and neglect the possible runoff caused by the emergence of the water table over the ground level. Thus, the thickness of the formation is large enough to accommodate for all the possible storages of water in the system, so that the water table does not intersect the upper face of the domain. The flow domain thus represents a simple unconfined aquifer; it is also assumed that the formation is uniform, that is the hydraulic properties (e.g. hydraulic conductivity) are constant over the entire domain. The numerical computations described in the following paragraphs have been carried out by varying the slope of the catchment.

We assume that the water flow is described by the Richards equation

\[
C(h) \frac{\partial h}{\partial t} = \frac{\partial}{\partial x_1} \left[ K(h) \left( \frac{\partial h}{\partial x_1} + 1 \right) \right] + q \quad (1)
\]

with \( C(h) = \frac{\partial \theta}{\partial h} \) the specific moisture content, \( h(x_1, t) \) the pressure head, \( K(h) \) the hydraulic conductivity of the material, \( \theta(x_1, t) \) the volumetric water content, \( q \) the source/sink term and \( x_1 \) the spatial coordinate vector.

For this analysis, we adopt the Brooks and Corey retention model (Brooks and Corey, 1964), that is \( S_c = (\theta - \theta_s)/(\phi - \theta_s) = (h_b/h)^\lambda \), with \( S_c \) the effective saturation, \( \theta_s \) the residual volumetric water content, \( \phi \) the porosity, \( h_b \) the bubbling or air entry pressure head and \( \lambda \) is the pore size distribution index. The specific moisture content \( C(h) \) therefore becomes

\[
C(h) = \frac{\partial \theta}{\partial h} = -(\phi - \theta_s) \lambda \frac{h_b^\lambda}{h^{\lambda+1}} \quad (2)
\]

The hydraulic conductivity of the material can be defined as a function of the saturated hydraulic conductivity \( K_s \), the bubbling pressure \( h_b \), the pressure head \( h \) and the pore size index \( \lambda \) \((n = 3 + 2/\lambda)\) as follows
In the following, we work with dimensionless variables because it helps in reducing the number of parameters governing the problem. The dimensionless form of the Richards equation can be derived by normalising the specific moisture content, the volumetric water content, the pressure head, the hydraulic conductivity of the material and the sink/source term using the appropriate scaling variables. Thus, we adopt the dimensionless variables

\[
K(h) = K_s \left( \frac{h_0}{h} \right)^{n_h}. \tag{3}
\]

Adopting the same approach, we split the boundary face into a Dirichlet boundary condition for the seepage face (in which the prescribed internal pressure \(p_0 \geq 0\)) and the Neumann condition for the region above the seepage face. The temporal variability of the length of the seepage face is automatically accounted for by the method. For details about the algorithm, the reader is referred to Chui and Freyberg (2007). The dimensionless storage and discharge were simulated and calculated directly from the model by incorporating the corresponding parameters used in Equation (7). We have followed different procedures for the steady-state and recession analysis. Brief procedures for each analysis will be introduced in the following sections.

EFFECT OF SOIL PARAMETERS

Before addressing the storage–discharge relationship, we carried out a set of computations to investigate the sensitivity of the results to changes in the soil parameters. The sensitivity analysis was carried out for some representative soil types from clay to sand, to cover most of the field soil ranges. Specifically, five soil types were chosen (the parameters are shown in Table I) to evaluate their effect on the results of the simulations. To speed up the computations and to quickly reach the stationary conditions (i.e. inflow equals outflow), initial conditions representing full saturation of the entire domain of area \((L^* L)\) were imposed, leading to a unit dimensionless storage. Constant dimensionless inward flux equal to \(Q/(LK_s) = 0.25\) was applied for the entire simulation. The effects of the parameters were analyzed for different values of the pore size distribution (\(n\)) and bubbling pressure \((h_b)\), keeping the other parameters constant. The dimensionless saturated storage \(S'\) and discharge \(Q'\) was calculated for the entire simulation. The results (Figure 2) show that changing the unsaturated parameters do not have a relevant effect in terms of nondimensional variables, on the discharge and storage, for both the transient and the steady-state conditions. The saturated storage calculated from the numerical model for five soil types as shown in Figure 2b experiences small variations in the volume of water stored in the system for soils having a clay property. This might be due to capillary fringe which, however, seems to have a minor effect on the results. The results were confirmed by other simulations with different configurations. Hence, in the

<table>
<thead>
<tr>
<th>Soil type</th>
<th>(h_b) (cm)</th>
<th>(n)</th>
<th>(\phi) (cm³/cm³)</th>
<th>(\theta_s) (cm³/cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sand soil</td>
<td>7.26</td>
<td>0.694</td>
<td>0.437</td>
<td>0.020</td>
</tr>
<tr>
<td>Loamy sand soil</td>
<td>8.69</td>
<td>0.553</td>
<td>0.437</td>
<td>0.035</td>
</tr>
<tr>
<td>Loam soil</td>
<td>11.15</td>
<td>0.252</td>
<td>0.463</td>
<td>0.027</td>
</tr>
<tr>
<td>Silt loam</td>
<td>20.76</td>
<td>0.242</td>
<td>0.501</td>
<td>0.015</td>
</tr>
<tr>
<td>Clay</td>
<td>37.30</td>
<td>0.165</td>
<td>0.475</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Source: Rawls et al. (1993).
following, we shall neglect the effect of the soil parameters on the dimensionless storage–discharge relation, and we shall use the parameters of loam soil for all the simulations.

**DERIVATION OF** $Q(S)$ **FROM A STEADY-STATE ANALYSIS**

As discussed in the Introduction, the use of storage–discharge relationship implies that the dynamics of the flow is approximated as a sequence of quasi-steady-state conditions, following the concept of kinematic modelling. Hence, it is a natural choice to determine the storage–discharge relationship through a series of steady-state numerical simulations. Because of the chosen set of dimensionless variables (Equation 4), and after neglecting the influence of the soil parameters (as discussed in the previous section), the simulation depends only on the slope and the dimensionless discharge $Q/\left(K_sL\right)$ which, at steady conditions, equals the inflow from the recharge.

We remark that the assumption of single-valued storage–discharge relation may not be valid for some cases, as emphasised in Sloan (2000) and Seibert et al. (2003).

Hence, we have performed a set of numerical simulations with different slopes and dimensionless recharge. Each steady-state simulation was carried out as described in the previous section by using full saturation initial conditions, imposing a given dimensionless recharge (equal to the desired dimensionless discharge), and performing a transient simulation until steady state is reached (the outflow is equal to the infiltration rate). The dimensionless saturated storage $S/((\phi - \theta_s)L^2)$ is recorded at the end of the simulations. The same procedure is repeated for several values of the dimensionless recharge and slope. As a final result, we have produced a set of six couples of $Q$ and $S$ for each slope. Each computation was carried out using a finite element mesh composed of 1598 triangular nonlinear elements, for a total of 3297 degrees of freedom. The model was run on a standard PC equipped with 4 GB of RAM and a dual-core CPU (2.90 GHz); each computation required approximately 3 h.

The numerical results at stationary conditions are displayed in Figure 3 (dots); each of the six panels refers to a specific slope. The relationship $Q(S)$ was derived by best fit of the numerical results. The results were best represented by a nonlinear, power–law relationship

$$\frac{Q}{K_sL} = a\left(\frac{S}{L^2(\phi - \theta_s)}\right)^b$$

(8)

where $a$ and $b$ are parameters which depend on the hillslope’s steepness. The findings seem to confirm the choice for a power law type of storage–discharge relation usually taken in the literature (e.g. Wittenberg and Sivapalan, 1999; Botter et al., 2010), as discussed in the Introduction. The fitted constant $b$ decreases from 1.4 (horizontal slope) to values below unity ($b=0.952$ for $30^\circ$ and $b=0.857$ for $45^\circ$). The exponent $b$ for the horizontal catchment is greater than unity and is comparable to the results of Lee (2007), who found that $b$ was 1.24 and 1.35 for flat ground surface and convex–concave topographic catchments, respectively. Other studies (e.g. Rupp and Selker, 2005, 2006; Harman and Sivapalan, 2009) have shown that the analytical solution of the nonlinear Boussinesq equation for sloping aquifers resulted in values of exponent $b$ ranging from less than 1 to 2. Field experiments by Wittenberg (1999) have shown the same results for exponent $b$ with average values of 2. Similarly, Chapman (1999) also derived values from 1.62 to 3.24 for 13 catchments in Australia. We emphasise that our scale (hillslope scale) is generally different and smaller than the scales considered in the other studies, and thus possible scale issues could arise. For those reasons, a perfect agreement between our exponent and those found at large scales is unlikely.

In turn, the constant $a$ displays a very small variability, regardless of the slope, in the range 0.72 and 0.74. Hence, we can further simplify the problem and assume a constant $a=0.73$. Doing so, the parameter $b$ can be easily...
related to the slope of the catchment by fitting the $b$ slope relation, obtaining $b = 0.5827\phi^2 - 1.163\phi + 1.408$, where $\phi$ is the slope of the catchment in radian. The curve is displayed in Figure 4.

In summary, the steady-state analysis has led to the identification of a power law storage–discharge relation for the assumed hillslope setup. We found that the power–law relation (Equation 8), written in dimensionless variables, can be applied to all the simulated cases with quite good agreement. All the approximations notwithstanding, the curves may be considered as a first, preliminary approach to estimate the $Q(S)$ relation as the function of a few relevant physical parameters, such as saturated hydraulic conductivity, effective porosity, length and steepness of the hillslope.

**Derivation of $Q(S)$ from recession curve analysis**

The recession curve is the specific part of the flood hydrograph after a rainfall event in which streamflow reduces. It is dominated by discharge of water from the subsurface during dry conditions and it helps to determine the retention characteristics of the basin and the subsurface storage. The study of the recession curve is also used to evaluate the volume of the dynamic storage of the subsurface. It is one of the most widely used methods to obtain the storage–outflow relationship curve. It is not our objective here to discuss the modelling approaches for recession analysis, but rather comparing the output relationship curve derived indirectly from recession analysis with the steady-state-derived one, as outlined in the previous section. In fact, in practice, it is impossible to measure the water storage $S$ in a hillslope. Conversely, the water discharge is easily measurable, and it is therefore much easier to determine $Q(S)$ from the recession curve analysis. The aim of this section is to check whether the two approaches (recession curve and steady-state analysis) lead to the same discharge–storage relation. The basic steps for the recession curve analysis

![Figure 4](image-url)
are briefly summarised here for the sake of clarity (further details can be found in Brutsaert and Nieber, 1977).

In the absence of recharge, the continuity equation in dimensionless form is written as

$$-\frac{dQ'}{dt} = \frac{dS'}{dt}.$$  \hspace{1cm} (9)

Adopting a power–law relation for $Q'(S') = aS'^b$, and substituting its inverse relation $S' = (Q'/a)^{1/b}$ in Equation (9), it yields

$$-\frac{dQ'}{dt} = a^{1/b}bQ'^{(\frac{b}{1+b})}.$$  \hspace{1cm} (10)

Thus, coefficients $a$ and $b$ can be derived from the analysis of the temporal derivative of discharge $Q(t)$ as a function of time, by fitting $-dQ'/dt$ versus $Q$ to Equation (10). The parameters of the conceptual storage model ($Q' = aS'^b$) can be calculated from the recession analysis. However, in the early stages, the slope of the recession curve might be greater than 2, resulting in a negative slope of the storage–discharge curve. Therefore, models for values of $\beta > 2$ must be recast to provide realistic values of discharge (Rupp and Woods, 2008; Clark et al., 2009). In our analysis, we never obtained negative values for $b$ as the entire recession curve was fitted, and not only in its early branch. In the following section, we compare the $a, b$ coefficients derived from the recession analysis to those presented in the previous section and derived from the steady-state analysis.

A series of numerical simulations were performed for that task. The initial condition is the steady-state configuration analyzed in the previous section for the highest recharge rates. Then, recharge is set to zero and free (unsteady) drainage is simulated. For each time step, we calculate the discharge $Q$ and the storage $S$. First, we checked that the relation $Q(S)$ derived by the unsteady, free drainage analysis was similar to that determined by the steady-state analysis. The comparison is shown in Figure 5, in which we plot the numerical results from the unsteady simulation (dots) with the $Q–S$ values obtained through the steady-state analysis (solid lines), performed in the previous section. The comparison suggests that the two types of analysis (steady and unsteady) provide similar results. Perhaps the differences are larger for the steepest case ($45^\circ$ slope), for which the formation seems to be drained faster than what is predicted by the steady-state model. This seems to suggest that for large slopes, dynamic effects may play a more significant role.

The previous comparison is based on the assumed availability of the storage $S$ (in our case, it is calculated from the numerical simulations). Thus, we follow the previously outlined recession analysis (which does not require $S$) and compare $dQ'/dt'$ as a function of $Q'$ with the prediction made in Equation (10) with $a, b$ given by the steady-state analysis (Figure 3). The results of the procedure are displayed in Figure 6, in which the dots represent the numerical results and the solid lines the prediction based on a steady-state analysis. Although $dQ'/dt'$ exhibits a noisy behaviour, a rather clear trend can be detected. The trend is fairly well predicted by the power law $Q(S)$ model elaborated in the steady-state analysis. Consistent with the results from Figure 5, the differences are larger for the higher slope. This exercise suggests that fitting the $a, b$ parameters through the standard recession analysis may lead to the correct identification of the $Q=AS^B$ structure. This finding is important because, in practice, $S$ is not measurable and the recession analysis is the only way to obtain information regarding the structure of the storage–discharge relation.

Figure 5. Comparison of the storage–discharge relation from a steady-state simulation with the same obtained by transient simulations. (a) Horizontal slope, (b) $5^\circ$ slope, (c) $10^\circ$ slope, (d) $20^\circ$ slope, (e) $30^\circ$ slope and (f) $45^\circ$ slope.
The recession analysis leads to a \( Q(S) \) relationship in hillslopes. The problem has been simplified to obtain a simple expression for \( Q(S) \), as the function of a few physical parameters. Because of the simplifications adopted (e.g. the shape of the hillslope, 2D computations and homogeneous soil properties) the derived solution cannot be compared with more complex analyses, for example those using distributed models. Still, our solution relaxes some of the simplifications usually adopted in the analytical models, such as those based on the Boussinesq formulation. Hence, our approach lies in between the simple analytical methods and the more complex distributed models.

By using a steady-state formulation of flow, which is consistent with the ‘kinematic’ nature of models using the storage–discharge relation, we have found that the dimensionless form of \( Q(S) \) is approximately a power law (Equation 8), which is often used in the hydrologic literature. The coefficient multiplying the power–law relation is almost constant. Conversely, the exponent depends mainly on the slope of the catchment, and a simple relation was found for it. The value of exponent \( n \) obtained for both flat and sloping catchments is comparable with the results of several numerical models and field experiments.

We have also simulated a recession curve analysis, which is the standard tool to infer the storage–discharge relation from measured discharge data. In fact, direct measurements of water storage in the hillslope are practically impossible. The recession analysis leads to a \( Q(S) \) relation which is very similar to that obtained using a steady-state analysis.

CONCLUSIONS

Numerical analysis has been used to obtain a better understanding on the formulation of storage–discharge \( Q(S) \) relationship in hillslopes. The problem has been simplified to obtain a simple expression for \( Q(S) \), as the function of a few physical parameters. Because of the simplifications adopted (e.g. the shape of the hillslope, 2D computations and homogeneous soil properties) the derived solution cannot be compared with more complex analyses, for example those using distributed models. Still, our solution relaxes some of the simplifications usually adopted in the analytical models, such as those based on the Boussinesq formulation. Hence, our approach lies in between the simple analytical methods and the more complex distributed models.

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We note again that, because of the (sometimes restrictive) assumptions adopted, the results of this study may not be universally valid, as actual hillslopes are complex and heterogeneous systems; the generalization of the proposed relations to more complex and realistic systems deserves further investigation. Still, the results presented here may serve as a preliminary tool for assessing the storage–discharge relation in a hillslope from physical parameters, such as the length and slope of the hillslope, saturated conductivity and effective porosity.

REFERENCES


