# EXERCISES, IV

Master of Science in Economics (MoSEc), LUISS, Rome:
Due on Friday, October 29, 2010

**Exercise 1** (Iso-elastic utility, I)**.** Consider the per-period utility $u : \mathbb{R}_{++} \to \mathbb{R}$ given by

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

where $\sigma > 0$. Under uncertainty, with complete markets, assuming a representative individual, compute equilibrium prices $\{R_{0,t}\}_{t=0}^{\infty}$. **Hint:** Notice that optimality imposes, for some Lagrange multiplier $\lambda > 0$,

$$\beta^t u'(c_t) = \lambda R_{0,t}.$$

**Exercise 2** (Iso-elastic utility, II)**.** Consider an economy with two individuals, $\mathcal{J} = \{1, 2\}$, having identical iso-elastic utilities (see exercise **??**) and identical subjective discount factors. Income of individual $i$ in $\mathcal{J}$ is $\{y_t^i\}_{t=0}^{\infty}$, while consumption is $\{c_t^i\}_{t=0}^{\infty}$. Under uncertainty, with complete markets, compute equilibrium prices $\{R_{0,t}\}_{t=0}^{\infty}$ and equilibrium consumptions. **Hint:** Market clearing now imposes

$$c_t^1 + c_t^2 = y_t^1 + y_t^2 = y_t,$$

where $\{y_t\}_{t=0}^{\infty}$ denotes the stochastic process for aggregate income. Exploit again Euler equations.

**Exercise 3** (Infinite-maturity asset)**.** Consider the per-period utility $u : \mathbb{R}_{+} \to \mathbb{R}$ given by

$$u(c) = \sqrt{c}.$$

Uncertainty is described by a Markov chain on states $\{\omega_1, \omega_2\}$. Given any current state, in the following period, the economy moves into one of the two states with equal probability. Income is $y(\omega_1)$ (respectively, $y(\omega_2)$) when economy is in state $\omega_1$ (respectively, $\omega_2$); an infinite-maturity security, in unitary net supply, pays off $d(\omega_1)$ (respectively, $d(\omega_2)$) when economy is in state $\omega_1$ (respectively, $\omega_2$). Compute the prices of the infinite-maturity security in each state (to simplify, assume that $d(\omega_1) = d(\omega_2) = 1$, $y(\omega_1) = 3$ and $y(\omega_2) = 8$).

**Exercise 4** (Negative risk-premium)**.** Under uncertainty, with complete markets, construct an example in which the return on an infinite-maturity security exhibits a negative risk premium at equilibrium.

**Exercise 5** (Yield curve)**.** Let $q_t^n$ be the price of a security paying off a unitary dividend only at maturity after $n$ periods, with $n = 1, 2, 3, \ldots$. Equivalently,

$$\rho_t^n = \frac{1 - q_t^n}{q_t^n}$$

is the risk-free net rate of interest after $n$ periods. Such securities are traded on the market from issuance up to the period before maturity ($n$ periods). According to the *pure expectations theory*,

$$q_t^2 = \left(\frac{1}{1+\rho_t^2}\right) = \left(\frac{1}{1+\rho_t^1}\right)\mathbb{E}_t\left(\frac{1}{1+\rho_{t+1}^1}\right) = q_t^1\mathbb{E}_t q_{t+1}^1$$

and, in general,

$$q_t^{n+1} = q_t^1\mathbb{E}_t q_{t+1}^1 \cdots \mathbb{E}_t q_{t+n}^1.$$

That is, the $n$-period (long-term) interest rate is determined by the expectations on one-period (short-term) interest rates. Show that, at equilibrium, this relation is in general violated. **Hint:** That formula does not account for the risk premium.