

# BASIC ASSET PRICING IN MACROECONOMICS

GAETANO BLOISE

Preliminary and incomplete version: October 7, 2010

**Warning, I:** These lecture notes are largely preliminary and incomplete. They should only be regarded as a complement to Chapters 8 (excluding sections 11-17) and 13 (excluding sections 14-16) of L. Ljungqvist and T.J. Sargent, *Recursive Macroeconomic Theory*, MIT Press, 2004.

**Warning, II:** The previous warning should be taken seriously.

## CONTENTS

1. Uncertainty	1
2. Consumption and Saving under Uncertainty	2
2.1. Euler Equation	2
2.2. Hall's Random Walk Theory of Consumption	3
2.3. Diversifiable labor income	4
2.4. Certainty equivalent	5
2.5. Precautionary saving	6
2.6. Liquidity constraints	8
2.7. Lucas' model of asset pricing	10
3. Equilibrium under Complete Markets	12
3.1. Arrow-Debreu equilibrium	12
3.2. Efficiency	13
3.3. Sequential equilibrium	14
3.4. Implications for asset pricing	16
4. No Arbitrage Restrictions	16
4.1. Stochastic discount factors	16
4.2. Implicit contingent claim prices	17
4.3. Fundamental values and speculative bubbles	18
5. Equilibrium Restrictions under Incomplete Markets	20
5.1. Optimal consumption	20
5.2. Euler equations	20
5.3. Equilibrium	21
5.4. Representative individual hypothesis	22
6. Empirical Content	23
6.1. Risk-aversion corrections	23
6.2. Martingale theory of stock prices	24
6.3. Capital asset pricing model	25
6.4. Equity premium puzzle	25
7. Intertemporal Budget Constraint and Optimality	26
7.1. Consolidation of budget constraints	26
7.2. Transversality condition under uniform impatience	27

## 1. UNCERTAINTY

The time horizon is  $\mathcal{T} = \{0, 1, 2, 3, \dots, t, \dots\}$ . Uncertainty is represented by a probability space,  $(S, \mathcal{F}, \mu)$ , and a filtration of  $\sigma$ -algebras,

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_t \subset \dots \subset \mathcal{F}.$$

To simplify and to avoid issues of integrability, it is assumed that  $\mathcal{F}_0$  is a trivial algebra and, for every  $t$  in  $\mathcal{T}$ ,  $\mathcal{F}_t$  is an algebra generated by a finite partition  $\mathcal{S}_t$  of  $S$ . An adapted plan  $(f_t)_{t \in \mathcal{T}}$  is simply a sequence of real-valued maps  $f_t : S \rightarrow \mathbb{R}$  such that, for every  $t$  in  $\mathcal{T}$ ,  $f_t$  is  $\mathcal{F}_t$ -measurable. Finally, for every measurable set  $E_t$  in the finite partition  $\mathcal{S}_t$  generating the  $\sigma$ -algebra  $\mathcal{F}_t$ ,  $\mu(E_t) > 0$ .

An alternative representation of uncertainty is provided by the event tree. For every  $t$  in  $\mathcal{T}$ , let  $\mathcal{S}_t$  be the finite partition of  $S$  generating the  $\sigma$ -algebra  $\mathcal{F}_t$ . Define the space of date-events

$$\mathcal{S} = \{(E_t, t) \in \mathcal{F} \times \mathcal{T} : E_t \in \mathcal{S}_t\}.$$

Every date-event  $\sigma$  in  $\mathcal{S}$  corresponds to a pair  $(E_\sigma, t_\sigma)$  in  $\mathcal{F} \times \mathcal{T}$ . Also, endow  $\mathcal{S}$  with the ordering

$$\tau \succeq \sigma \text{ if and only if } E_\tau \subset E_\sigma \text{ and } t_\sigma \leq t_\tau.$$

This ordering establishes that date-event  $\tau$  in  $\mathcal{S}$  succeeds to date-event  $\sigma$  in  $\mathcal{S}$ , meaning that it occurs at a later date ( $t_\sigma \leq t_\tau$ ) and, possibly, some additional information is acquired ( $E_\tau \subset E_\sigma$ ). Let  $\phi$  in  $\mathcal{S}$  be the initial date-event (that is,  $\sigma \succeq \phi$  for every  $\sigma$  in  $\mathcal{S}$ ) and, for every date-event  $\sigma$  in  $\mathcal{S}$ , let

$$\sigma_+ = \{\tau \in \mathcal{S} : t_\tau = t_\sigma + 1\}$$

be the finite set of all immediate successors of that date-event, where

$$\mathcal{S}_\sigma = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}.$$

An adapted plan  $(f_t)_{t \in \mathcal{T}}$  can be equivalently represented as

$$((f_\sigma)_{\sigma \in \mathcal{S}_t})_{t \in \mathcal{T}} = (f_\sigma)_{\sigma \in \mathcal{S}}.$$

Indeed, at every  $\sigma$  in  $\mathcal{S}$ , by  $\mathcal{F}_{t_\sigma}$ -measurability,  $f_\sigma = f_{t_\sigma}(s)$  for every  $s$  in  $E_\sigma$ .

These two representations of uncertainty are equivalent. The latter requires a finite set of events in every period, but it is more treatable. The former is consistent with any stochastic structure, but issues of integrability should be properly taken into account when the hypothesis of finitely many events in every period is removed.

For practical purposes, it is convenient to consider the following binomial representation of uncertainty. In every period of trade  $t$  in  $\mathcal{T}$ , the economy might be either in expansion (the current state is  $u$ ) or in recession (the current state is  $d$ ), with some initial predefined state. Thus,  $S$  contains all infinite sequences of elements of  $\{u, d\}$ , beginning from the predefined initial state in  $\{u, d\}$ . Also,  $\mathcal{S}$  is the set of all truncated sequences in  $S$  and  $\mathcal{S}_t$  is the set of all sequences in  $S$  truncated at period  $t$  in  $\mathcal{T}$ . An adapted plan  $f = (f_t)_{t \in \mathcal{T}}$  might be simply regarded as a map  $f : \mathcal{S} \rightarrow \mathbb{R}$ , where  $f(s_0, s_1, s_2, \dots, s_{t-1}, s_t)$  is its value at the partial history  $(s_0, s_1, s_2, \dots, s_{t-1}, s_t)$  of elements of  $\{u, d\}$ , given the predefined initial state in  $\{u, d\}$ .

## 2. CONSUMPTION AND SAVING UNDER UNCERTAINTY

**2.1. Euler Equation.** Consider a single consumer with preferences over uncertain consumption represented, in every period of trade  $t$  in  $\mathcal{T}$ , by

$$U_t(c) = \mathbb{E}_t \sum_{s \in \mathcal{T}} \beta^s u(c_{t+s}).$$

Here, expectation is conditional on information available in period of trade  $t$  in  $\mathcal{T}$ . As usual,  $1 > \beta > 0$  denotes the discount factor and  $\varrho = \beta^{-1} - 1 > 0$  is the rate of time preference. The per-period utility  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is smoothly strictly increasing and smoothly strictly concave. It characterizes the degree of intertemporal substitution, as in the case of no uncertainty. However, it now also reflects the attitude of the consumer toward risk. Importantly, these two features cannot be properly separated in utilities with this time additively separable form. More precisely,

$$\text{coefficient of relative risk aversion} = -\frac{u''(c)c}{u'(c)}$$

and

$$\text{elasticity of intertemporal substitution} = -\frac{u'(c)}{u''(c)c}.$$

The consumer is assumed to be uncertain about both future labor income and the returns on assets. To simplify, I here restrict attention to investment in a single asset, so avoiding issues related to portfolio theory. Consistently, the budget constraint is given, in every period  $t$  in  $\mathcal{T}$ , by

$$s_{t+1} = (1 + r_{t+1})(s_t + y_t - c_t),$$

where  $s_t$  is financial wealth at the beginning of the period,  $y_t$  is labor income and  $c_t$  is consumption. The term  $(s_t + y_t - c_t)$  is gross saving, that is, the excess of financial wealth and income over consumption. As a first approximation, I neglect debt limits, which should be imposed in order to enforce solvency when debt is allowed. Notice that the (net) rate of return  $r_{t+1}$  of the risky asset is uncertain in period  $t$  in  $\mathcal{T}$ .

An interior optimal consumption plan satisfies the Euler equation, that is, in every period  $t$  in  $\mathcal{T}$ ,

$$\beta \mathbb{E}_t u'(c_{t+1})(1 + r_{t+1}) = u'(c_t).$$

The validity of this simple first-order condition is proved by means of a canonical variational argument. Consider a slight budget-balanced substitution of current consumption for future consumption. The impact on expected utility of this slight variation is given by

$$\Phi_t(\epsilon_t) = u(c_t - \epsilon_t) + \beta \mathbb{E}_t u(c_{t+1} + (1 + r_{t+1})\epsilon_t),$$

where  $\epsilon_t$  is sufficiently small. When  $\epsilon_t > 0$ , this corresponds to a slight increase in gross savings; when  $\epsilon_t < 0$ , it is a slight increase in current debt (and, hence, it is assumed that debt limits are not binding). At an optimal consumption plan, this budget-feasible slight variation in consumption cannot increase expected utility, that is,

$$\Phi_t(\epsilon_t) \leq \Phi_t(0).$$

Hence, the Euler equation needs be satisfied. Notice that, when the returns on assets involve no uncertainty, conditional on information available in period  $t$  in  $\mathcal{T}$ , the Euler equation becomes

$$\beta (1 + r_t) \mathbb{E}_t u' (c_{t+1}) = u' (c_t),$$

when now  $r_t$  denotes the certain (net) rate of return on assets from period  $t$  to period  $t + 1$ .

Using a second-order approximation,

$$u' (c_{t+1}) \approx u' (c_t) + u'' (c_t) (c_{t+1} - c_t).$$

Introducing the rate of growth of consumption,

$$g_{t,t+1} = \frac{c_{t+1} - c_t}{c_t},$$

the approximate Euler equation becomes

$$\mathbb{E}_t r_{t+1} = \theta + \gamma (c_t) \mathbb{E}_t g_{t,t+1} (1 + r_{t+1}),$$

where

$$\gamma (c) = - \frac{u'' (c) c}{u' (c)}$$

and

$$\theta = \frac{1 - \beta}{\beta}.$$

**2.2. Hall's Random Walk Theory of Consumption.** Suppose that interest rate is constant over time and equal to the rate of time preference, that is,  $\beta (1 + r) = 1$ . Euler equation reduces to

$$\mathbb{E}_t u' (c_{t+1}) = u' (c_t).$$

This is a basic observation asserting that the marginal utility of consumption follows an univariate first-order Markov process and that no other variables in the information set help to predict (or to Granger cause)  $u' (c_{t+1})$ , once lagged  $u' (c_t)$  has been included.

As an example, with the constant relative risk aversion utility,

$$u (c) = \frac{c^{1-\gamma}}{1-\gamma}, \text{ with } \gamma > 0,$$

Euler equation becomes

$$\mathbb{E}_t c_{t+1}^{-\gamma} = c_t^{-\gamma}.$$

Using aggregate time series, Hall tested implications for the special case of quadratic utility.

**2.3. Diversifiable labor income.** Consider the case of a fully diversifiable labor income risk. Even though there is no market in human capital, individuals might be able to protect against part of their labor income uncertainty by holding portfolios of securities whose returns are negatively correlated with labor income. A complete diversification is certainly not empirically plausible. Thus, this should only be treated as a sort of benchmark.

Fully diversification can be represented, without loss of generality, as no labor income, as, in a sense, all income is derived from tradable wealth. Also, to obtain an explicit solution, assume that per-period utility is of the form

$$u (c) = \log c.$$

This exhibits constant elasticity of intertemporal substitution and, furthermore, it bears very peculiar implications on optimal consumption. The Euler equation becomes

$$(*) \quad \frac{1}{c_t} = \beta \mathbb{E}_t (1 + r_{t+1}) \frac{1}{c_{t+1}}.$$

The budget constraint is

$$(**) \quad s_{t+1} = (1 + r_{t+1})(s_t - c_t).$$

To obtain a solution, postulate that optimal consumption is a constant share of financial wealth, that is,

$$c_t = \mu s_t.$$

Exploiting budget constraint (\*\*), one obtains

$$\frac{1}{s_t} = (1 - \mu)(1 + r_{t+1}) \frac{1}{s_{t+1}}.$$

This straightforwardly implies

$$\frac{1}{c_t} = (1 - \mu)(1 + r_{t+1}) \frac{1}{c_{t+1}}.$$

Taking the expected value, the Euler equation is satisfied if and only if  $\mu = (1 - \beta)$ . This proves that optimal consumption obeys the linear rule

$$c_t = (1 - \beta) s_t.$$

Thus, under the assumption that labor income is fully diversifiable and that utility is logarithmic, the characterization of consumption is identical to that under certainty. The marginal propensity to consume out of wealth is approximately equal to the rate of time preference, so that saving and consumption decisions only depend on the rate of time preference, and not on financial variables. Changes in future incomes, or dividends, or interest rates, affect consumption through their effect on wealth.

**2.4. Certainty equivalent.** I here assume that utility is quadratic and that the only traded asset is risk-less. No restrictions are imposed on uncertain labor income. Thus, per-period utility is of the form

$$u(c) = \frac{1}{2} (ac - bc^2),$$

where  $a > 0$  and  $b > 0$  are parameters. Furthermore, the budget constraint can be written as

$$s_{t+1} = (1 + r_t)(s_t + y_t - c_t),$$

reflecting the fact that the return on investment is not uncertain. For convenience, also assume that the rate of interest is constant and satisfies

$$\beta(1 + r_t) = \beta(1 + r) = 1.$$

This restriction is, in a sense, necessary, as a persistent discrepancy between market interest rate and the rate of time preference would produce permanently increasing, or permanently decreasing, consumption over time. This feature, which is delivered by the hypothesis of constant rate of time preference, seems rather unappealing.

It is simple to verify that the Euler equation takes the form

$$c_t = \mathbb{E}_t c_{t+1}.$$

In other terms, optimal consumption is expected to be constant over time. Consolidating the budget constraint, one obtains

$$\lim_{t \rightarrow \infty} \left( \frac{1}{1+r} \right)^t s_{t+1} + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t c_t = s_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t y_t.$$

Notice that this is an integration of realized budget constraints. Also, by the transversality condition, the first term on the left hand-side vanishes, for otherwise the individual would be over-accumulating assets across time. Hence, taking the expected value, one obtains

$$\sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t \mathbb{E}_0 c_t = s_0 + \mathbb{E}_0 \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t y_t.$$

Using the Euler equation, jointly with the law of iterated expectation,

$$c_0 = \mathbb{E}_0 c_t.$$

Therefore,

$$c_0 = \left( \frac{r}{1+r} \right) \left( s_0 + \mathbb{E}_0 \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t y_t \right).$$

And, by similar arguments, at any other period  $t$  in  $\mathcal{T}$ ,

$$c_t = \left( \frac{r}{1+r} \right) \left( s_t + \mathbb{E}_t \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s y_{t+s} \right)$$

Optimal consumption exhibits the certainty equivalent property. (This is a common feature of all quadratic problems.) In other terms, the solution is identical to that which would obtain in the case of no uncertainty. Consumption is a linear function of total wealth, which is the sum of financial wealth and the present value of expected labor income. The marginal propensity to consume out of wealth is (approximately) equal to the interest rate and, hence, to the rate of time preference.

A complete characterization of optimal consumption allows for studying the effect of the volatility of labor income on consumption, that is, how consumption smooths or amplifies movements in income. The marginal propensity to consume out of labor income depends on the persistence of the process for labor income. Indeed, exploiting the above characterization,

$$c_t - c_{t-1} = \left( \frac{r}{1+r} \right) \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s (\mathbb{E}_t y_{t+s} - \mathbb{E}_{t-1} y_{t+s}).$$

Changes in consumption depend on the present value of revisions in future labor income.

As a first example, suppose that income disturbances are identically and independently distributed over time, that is,

$$y_{t+1} = \bar{y} + \epsilon_{t+1}.$$

Here,  $\bar{y}$  is the permanent part of current income and  $\epsilon_t$  is the transitory part of current income, with  $\mathbb{E}_t \epsilon_{t+1} = 0$ . By the above formula, changes in consumption obey the rule

$$c_t - c_{t-1} = \left( \frac{r}{1+r} \right) \epsilon_t.$$

Thus, consumption only reacts mildly in response to a temporary shock on income.

As a second example, suppose that income is given by a random walk,

$$y_{t+1} = y_t + \epsilon_{t+1}.$$

The innovation on income is now permanent. Changes in consumption satisfy

$$c_t - c_{t-1} = \epsilon_t.$$

A permanent shock on income completely translates into an adjustment in consumption

Finally, assume that labor income follows an autoregressive process of first order, that is,

$$y_{t+1} = \rho y_t + \epsilon_{t+1},$$

where  $0 < \rho < 1$ . The effects on consumption are given by

$$c_t - c_{t-1} = \left(\frac{r}{1+r}\right) \sum_{s=0}^{\infty} \left(\frac{\rho}{1+r}\right)^s \epsilon_t.$$

Thus, the change in consumption in reaction to a persistent (but not permanent) shock lies qualitatively in between that induced by a purely transitory shock and that induced by a permanent shock. The marginal propensity to consume is given by

$$\left(\frac{r}{1+r}\right) < \left(\frac{r}{1+r-\rho}\right) < 1.$$

**2.5. Precautionary saving.** Certainty equivalence explicitly rules out precautionary saving. Variations in consumption only depend on conditional expectation of future income, independently of conditional variance. Hence, a change in the uncertainty of future labor income (with unchanged mean) leaves consumption choices unchanged under certainty equivalence.

To understand the role of prudence, consider a constant interest rate coinciding with the rate of time preference. Euler equation yields

$$\mathbb{E}_t u'(c_{t+1}) = u'(c_t).$$

So long as consumers are risk-averse ( $u'' > 0$ ), an increase in the variance of consumption negatively affects expected utility. However, the effect of additional uncertainty on optimal consumption depends on its impact on Euler equation and, hence, on marginal utilities. Under certainty equivalence (quadratic utility), marginal utilities are linear and, thus, an increase in the variance of future consumption leaves expected marginal utility unaltered. In a more plausible case, marginal utilities are convex ( $u''' > 0$ ). Therefore, an increase in uncertainty raises the expected marginal utility. To satisfy Euler equation, this increase of future expected marginal utility needs to be balanced by an increase in current marginal utility and, so, by a decrease in current consumption. In other terms, additional uncertainty induces to defer consumption by increasing savings for precautionary purposes. This explains the effect of prudence ( $u''' > 0$ ) on precautionary saving.

For a better understanding of the role of prudence in inducing precautionary saving, assume an interest rate coinciding with the rate of time preference. Consider the following comparative statics exercise: beginning from no uncertainty, future income is now subject to a small random shock  $\epsilon$ , with  $\mathbb{E}\epsilon = 0$ ; this induces a readjustment  $\Delta s$  in current saving; by Euler equation, changes satisfy

$$\mathbb{E} u'(c + (1+r)\Delta s + \epsilon) = u'(c - \Delta s).$$

Assuming that marginal utility is convex, by Jensen's Inequality,

$$u'(c + (1+r)\Delta s) < u'(c - \Delta s).$$

As marginal utility is decreasing,  $\Delta s > 0$ . Assuming that marginal utility is concave, by Jensen's Inequality,

$$u'(c + (1+r)\Delta s) > u'(c - \Delta s).$$

As marginal utility is decreasing,  $\Delta s < 0$ .

More in general, supposing that marginal utility is convex, Euler equation implies

$$u'(\mathbb{E}_t c_{t+1}) \leq u'(c_t).$$

Therefore, as marginal utility is also decreasing,

$$\mathbb{E}_t c_{t+1} \geq c_t$$

and, using the law of iterated expectation,

$$\mathbb{E}_t c_{t+s} \geq c_t.$$

Define  $\psi_{t,t+s} = c_{t+s} - c_t$  and observe that

$$\mathbb{E}_t \psi_{t,t+s} = \mathbb{E}_t c_{t+s} - c_t \geq 0.$$

Consolidating budget constraints and using transversality,

$$\mathbb{E}_t \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s c_{t+s} = s_t + \mathbb{E}_t \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s y_{t+s},$$

which finally yields

$$c_t = \left( \frac{r}{1+r} \right) \left( s_t + \mathbb{E}_t \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s y_{t+s} - \mathbb{E}_t \sum_{s=0}^{\infty} \left( \frac{1}{1+r} \right)^s \psi_{t,t+s} \right).$$

The last term is the effect produced by precautionary saving with respect to the certainty equivalent principle. Notice that an increase in variability of consumption in the remote future induces a contraction of current consumption, as savings increase immediately for precautionary purposes.

Alternatively, notice that

$$\left( \frac{1}{1+r} \right) \mathbb{E}_t u'(c_{t+1}) s_{t+1} + u'(c_t) c_t = u'(c_t) s_t + u'(c_t) y_t.$$

To explore more direct implications, consider the case of a constant elasticity of substitution,

$$\mathbb{E}_t c_{t+1}^{-\gamma} = c_t^{-\gamma}.$$

Taking a first-order approximation around consumption  $c_t$ ,

$$\mathbb{E}_t \left( \frac{c_{t+1} - c_t}{c_t} \right) = \left( \frac{1+\gamma}{2} \right) \mathbb{E}_t \left( \frac{c_{t+1} - c_t}{c_t} \right)^2.$$

This permits two basic remarks. First, the consumption decision do not obey certainty equivalence. Prudence induces a precautionary savings motive. Uncertainty about future consumption, captured in the second term of the left hand-side of above equation, tilts the consumption profile upward in expectation, relative to the certainty case. Thus, consumption growth is higher in the uncertainty case than in the certainty case, meaning that individuals tend to postpone consumption for precautionary motives. Second, the degree to which consumption is postponed (*i.e.*,

the degree to which there is precautionary saving) as reaction to future uncertainty is determined by the parameter controlling prudence. In particular, increases with the so-called index of relative prudence,

$$\text{index of relative prudence} = -\frac{u'''(c)c}{u''(c)} = 1 + \gamma.$$

**2.6. Liquidity constraints.** So far I neglected the impact of borrowing limits on optimal consumption, allowing for an arbitrarily large amount of debt, under no Ponzi game restrictions. Borrowing constraints seem empirically plausible and formal econometric tests seem to indicate not only their presence, but also their effect on consumption. I now explicitly assume that there exist borrowing constraints and, for simplicity, that individuals cannot borrow at all. The budget constraint,

$$s_{t+1} = (1 + r_{t+1})(s_t + y_t - c_t),$$

is complemented by a no borrowing restriction of the form

$$s_{t+1} \geq 0.$$

To understand the impact of borrowing limits on Euler equation, consider again a slight substitution of current consumption for future consumption. The variation in expected utility is given by

$$\Phi_t(\epsilon_t) = u(c_t - \epsilon_t) + \beta \mathbb{E}_t u(c_{t+1} + (1 + r_{t+1})\epsilon_t).$$

However, as debt is not allowed, the readjustment is subject to the additional restriction

$$s_t + \epsilon_t \geq 0.$$

Optimality of the consumption plan implies that this substitution cannot produce a welfare increase and, hence, by first-order conditions,

$$\beta \mathbb{E}_t (1 + r_{t+1}) u'(c_{t+1}) \leq u'(c_t)$$

and

$$\beta \mathbb{E}_t (1 + r_{t+1}) u'(c_{t+1}) = u'(c_t) \text{ if } s_t > 0.$$

This adjusted Euler equation is the basis for empirical tests for liquidity constraints.

Though any issue of insurance is vague under no uncertainty, it is nonetheless interesting to characterize optimal consumption in the case of certainty, when interest rate coincides with the rate of time preference,  $\beta(1 + r) = 1$ . Preliminarily observe that, by Euler equation, as marginal utilities are decreasing,

$$c_{t+1} \geq c_t.$$

Also, by consolidation of budget constraints,

$$\left(\frac{1}{1+r}\right)^{k+1} s_{t+k+1} + \sum_{s=0}^k \left(\frac{1}{1+r}\right)^s c_{t+s} = s_t + \sum_{s=0}^k \left(\frac{1}{1+r}\right)^s y_{t+s};$$

and, using debt limits,

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} \leq s_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_{t+s}.$$

Thus, optimal consumption is weakly increasing over time. Define

$$\bar{c} = \sup_t \bar{c}_t = \sup_t \left(\frac{r}{1+r}\right) \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_{t+s}.$$

This would be the constant consumption chosen in the absence of borrowing constraints. It turns out that, when there are no initial claims,  $s_0 = 0$ ,

$$c = \lim_{t \rightarrow \infty} c_t = \sup_t \bar{c}.$$

This shows that the impact of borrowing constraints on optimal consumption does not vanish until the annuity of the remainder of the income process is maximized.

To prove this claim, suppose that  $c > \bar{c}$  and, thus,  $c_t \geq \bar{c} + \epsilon$  for every sufficiently large  $t$ . By budget constraint,

$$\begin{aligned} \left(\frac{1+r}{r}\right) \bar{c} + \left(\frac{1+r}{r}\right) \epsilon &\leq \\ \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} &\leq s_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_{t+s} \\ &\leq s_t + \left(\frac{1+r}{r}\right) \bar{c}. \end{aligned}$$

Furthermore, it can be assumed that borrowing limits are binding at  $t$ ,  $s_t = 0$ , so producing a contradiction. Suppose that  $c < \bar{c}$  and, thus,  $c_t < \bar{c} - \epsilon$  for every  $t$ . Therefore, there exists  $t$  such that  $c_{t+s} < \bar{c}_t - \epsilon$  for every  $s$ . Hence, for any sufficiently large  $k$ ,

$$\sum_{s=0}^k \left(\frac{1}{1+r}\right)^s c_{t+s} < s_t + \sum_{s=0}^k \left(\frac{1}{1+r}\right)^s y_{t+s} - \epsilon;$$

and, by budget constraints,

$$s_{t+k+1} = (1+r)^{k+1} \left( s_t - \sum_{s=0}^k \left(\frac{1}{1+r}\right)^s (c_{t+s} - y_{t+s}) \right) \geq (1+r)^{k+1} \epsilon.$$

The individual is clearly over-accumulating assets, so violating optimality.

Moving to the case of uncertainty, If the certainty equivalence solution has associated positive asset holdings, then it is the optimal consumption allocation even in the presence of borrowing constraints. Whether or not this is the case obviously depends crucially on the income process. If asset holdings follow a random walk, as it happens with identically and independently distributed income shocks, this assumption is obviously violated. Interestingly, even for quadratic utility, risk-aversion induces to postponing consumption to protect against negative future income shocks. For quadratic utilities, Euler equation takes the form

$$\mathbb{E}_t c_{t+1} \geq c_t.$$

Thus, iterating,

$$\mathbb{E}_{t+s} c_{t+s} \geq c_t.$$

Invoking budget constraints and transversality,

$$c_t = \left(\frac{r}{1+r}\right) \left( s_t + \mathbb{E}_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_{t+s} - \mathbb{E}_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \psi_{t,t+s} \right),$$

where

$$\mathbb{E}_t \psi_{t,t+s} = \mathbb{E}_t c_{t+s} - c_t \geq 0.$$

This simple transformation shows that, in empirical investigation, it is rather to separate the effects of borrowing constraints from those produced by precautionary saving.

**2.7. Lucas' model of asset pricing.** Suppose that the only asset is a share of an enterprise. The price of a share is  $q_t$ , measured in terms of consumption in period  $t$ , and a share entitles to random positive dividends  $y_t$ , in terms of consumption, at the beginning of period  $t$ . Under this interpretation,  $s_t$  might be interpreted as the quantity of shares owned by the consumer at the beginning of period  $t$ . Thus, the uncertain return on savings is

$$1 + r_{t+1} = \frac{q_{t+1} + d_{t+1}}{q_t}.$$

Euler equation yields

$$\beta \mathbb{E}_t u'(c_{t+1}) (q_{t+1} + d_{t+1}) = u'(c_t) q_t.$$

This is only an optimality restriction and does not bear any insight into the equilibrium determination of prices.

Lucas supposes an economy consisting of a large number of identical individuals with now individual income (that is, with a completely diversified risk on labor income). Resources for consumption are only provided by dividends distributed by the enterprise. (In Lucas' metaphor, the enterprise is simple a tree and dividends are fruits.) Thus, equilibrium requires

$$c_t = d_t = y_t.$$

This captures in an essential way the logic of general equilibrium and provides a basic computable framework for the understanding of price determination at equilibrium.

Optimality of consumption plans, through Euler equation, now requires

$$u'(y_t) q_t = \beta \mathbb{E}_t u'(y_{t+1}) (q_{t+1} + y_{t+1}).$$

Solving recursively and using the law of iterated expectation,

$$q_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left( \frac{u'(y_{t+s})}{u'(y_t)} \right) y_{t+s}.$$

The share price is an expected discounted stream of dividends evaluated by time-varying and stochastic discount rates. Such discount rates are determined by dividends. Suppose, for instance, that utility is logarithmic,  $u(c) = \log c$ . The pricing formula becomes

$$q_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s y_{t+s}$$

or, equivalently,

$$q_t = \left( \frac{\beta}{1 - \beta} \right) y_t.$$

Notice that the price of the stock only depends on the current state of the economy.

In general, suppose that dividends are determined only by some observable state  $s$  in some state space  $S$ . The evolution of states is governed by some Markov process with a time-invariant probability distribution. The conditional expectation will eventually be defined with respect to this transition probability.

Supposing the current price only depend on the current state of the economy, the Euler equation becomes

$$q(s) u'(y(s)) = \beta \mathbb{E}_s u'(y(s')) (q(s') + y(s'))$$

or

$$(*) \quad w(s) = \beta \mathbb{E}_s w(s') + g(s'),$$

where  $w(s) = q(s) u'(y(s))$  and

$$g(s) = \beta \mathbb{E}_s y(s') u'(y(s')).$$

The latter term is bounded if per-period utility is bounded, as

$$u'(c) c \leq u(c) - u(0).$$

Functional equation (\*) admits a unique solution.

For a finite Markov process, restriction (\*) might be written as

$$w(s) = \beta \sum_{s' \in S} w(s') \pi(s, s') + g(s)$$

or, in matrix terms,

$$w = \beta \Pi w + g,$$

where  $\Pi$  is the stochastic matrix representing the Markov chain. This admits a unique solution,

$$w = (I - \beta \Pi)^{-1} g.$$

### 3. EQUILIBRIUM UNDER COMPLETE MARKETS

**3.1. Arrow-Debreu equilibrium.** The fundamental assumption is that individuals can trade to hedge against individual income uncertainty. We suppose that there is a complete set of contingent consumption claims that are traded at the initial period, before any uncertainty has been revealed. Thus, every individual is subject to a single intertemporal budget constraint of the form

$$\mathbb{E}_0 \sum_{t=0}^{\infty} p_t c_t^i \leq p_0 s_0^i + \mathbb{E}_0 \sum_{t=0}^{\infty} p_t y_t^i,$$

where  $p_t$  represents the price of consumption to be delivered in period  $t$  conditional on revealed information. This construction is admittedly abstract and should be regarded as as benchmark.

An equilibrium is defined as a collection of consumption plans  $(c^i)_{i \in N}$  and prices  $p$  such that (a) every consumption plan is optimal, for the individual, subject to the intertemporal budget constraint, and (b) markets clear, that is,

$$\sum_{i \in N} c_t^i = \sum_{i \in N} y_t^i = y_t.$$

Notice that, at equilibrium, all individuals exhaust the intertemporal budget, as utility is increasing in consumption. Hence,

$$\frac{1}{p_0} \mathbb{E}_0 \sum_{t=0}^{\infty} p_t \sum_{i \in N} (c_t^i - y_t^i) = s_0^i.$$

Aggregating across individuals and exploiting feasibility, it follows that

$$\sum_{i \in N} s_0^i = 0.$$

Hence, an equilibrium exists only for a balanced distribution of initial wealth. This is related to the inexistence of outside money and speculative bubbles at equilibrium under complete markets.

Optimality of consumption for individuals implies a first-order condition, at an interior optimum, of the form

$$\beta^t u'(c_t^i) = \lambda^i p_t,$$

where  $\lambda^i$  denotes the Lagrange multiplier. Denoting  $\phi$  the inverse of marginal utility, this condition becomes

$$c_t^i = \phi(\lambda^i \beta^{-t} p_t).$$

Also, by market clearing,

$$\sum_{i \in N} \phi(\lambda^i \beta^{-t} p_t) = y_t.$$

This reveals an important feature of equilibrium: prices only depend on the aggregate endowment and, hence, consumptions of individuals only depend on the aggregate endowment.

To make this more transparent, suppose that aggregate endowment is non-stochastic and constant over time at level  $\bar{y}$ . As marginal utilities are strictly decreasing, each  $\phi$  is strictly monotone. Hence, prices fulfil

$$p_{t+1} = \beta p_t.$$

Also, all consumptions are constant over time. Notice that this happens even though incomes of individuals are truly stochastic.

Suppose that aggregate endowment follows a Markov process over some finite state space  $S$ . Thus,

$$y_t(s) = y(s),$$

where  $s$  denotes the current realization of the shock. Equilibrium prices satisfy

$$p_t = p_t(s) = \beta^t \psi(s),$$

where  $\psi(s)$  is the unique solution to

$$\sum_{i \in N} \phi(\lambda^i \psi(s)) = y(s).$$

Consumptions fulfil

$$c_t^i = c^i(s) = \phi(\lambda^i \psi(s)).$$

Thus, equilibrium variables also follow a Markov process.

For instance, under constant elasticity of intertemporal substitution,  $u'(c) = c^{-\gamma}$  and  $\phi(v) = v^{-\frac{1}{\gamma}}$ . Hence, equilibrium requires

$$p_t = \beta^t \left( \frac{1}{\sum_{i \in N} (\lambda^i)^{-\frac{1}{\gamma}}} \right)^{-\gamma} y_t^{-\gamma}$$

and

$$c_t^i = \left( \frac{(\lambda^i)^{\frac{1}{\gamma}}}{\sum_{i \in N} (\lambda^i)^{\frac{1}{\gamma}}} \right) y_t = \mu^i y_t.$$

Consumptions are perfectly correlated with aggregate endowment. Finally, using the intertemporal budget constraint

$$\mu^i = \left( \mathbb{E}_0 \sum_{t=0}^{\infty} p_t y_t \right)^{-1} \left( p_0 s_0 + \mathbb{E}_0 \sum_{t=0}^{\infty} p_t y_t^i \right).$$

This also shows that, for some distribution of aggregate income and initial wealth, an equilibrium might be constructed for every distribution of shares  $(\mu^i)_{i \in N}$ .

**3.2. Efficiency.** Equilibrium is efficient under complete markets, as all risk-sharing opportunities are exploited. In other terms, risk is efficiently distributed across individuals. This is the so-called First Welfare Theorem. Indeed, assume that equilibrium allocation  $(c^i)_{i \in N}$  is strictly Pareto dominated by an alternative feasible allocation  $(\bar{c}^i)_{i \in N}$ . By optimality of consumption plans at equilibrium,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} p_t \bar{c}_t^i > p_0 s_0^i + \mathbb{E}_0 \sum_{t=0}^{\infty} p_t y_t^i.$$

Hence, aggregating across individuals,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} p_t \sum_{i \in N} \bar{c}_t^i > \mathbb{E}_0 \sum_{t=0}^{\infty} p_t \sum_{i \in N} y_t^i.$$

(Indeed, if not, the individual would have chosen the alternative consumption plan, violating optimality at equilibrium.) Feasibility, however, imposes

$$\sum_{i \in N} \bar{c}_t^i \leq \sum_{i \in N} y_t^i.$$

Therefore, as prices are positive,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} p_t \sum_{i \in N} \bar{c}_t^i \leq \mathbb{E}_0 \sum_{t=0}^{\infty} p_t \sum_{i \in N} y_t^i,$$

so yielding a contradiction. The Second Welfare Theorem establishes that any efficient allocation can be sustained as an equilibrium for some balanced distribution of initial claims.

**3.3. Sequential equilibrium.** Trade occurs now sequential in every period, conditional on revealed information. Markets are (sequentially) complete: there exist financial contracts in every period for the delivery of one unit of consumption in the following period for every realized event. These contracts are often called elementary Arrow securities, contingent claims or (one-period) insurance contracts. The budget constraint is of the form

$$\mathbb{E}_t m_{t,t+1} s_{t+1}^i + (c_t^i - y_t^i) \leq s_t^i,$$

where  $m_{t,t+1}$  denotes the price of elementary Arrow securities. As debt is permitted, there is also a solvency requirement of the form

$$-f_{t+1}^i \leq s_{t+1}^i,$$

where  $f_t^i$  denotes the maximum amount of debt. In particular, natural debt limits take the form

$$f_t^i = \frac{1}{p_t} \mathbb{E}_t \sum_{s=0}^{\infty} p_{t+s} y_{t+s}^i,$$

as in the case of certainty. They require that individuals be able to repay outstanding debt in a finite number of periods.

It can be verified that, at an interior optimum, Euler equation takes the form

$$(\dagger) \quad m_{t,t+1} u'(c_t^i) \geq \beta u'(c_{t+1}^i)$$

and

$$(\ddagger) \quad u'(c_t^i) \mathbb{E}_t m_{t,t+1} (s_{t+1}^i + f_{t+1}^i) = \beta \mathbb{E}_t u'(c_{t+1}^i) (s_{t+1}^i + f_{t+1}^i).$$

Notice that, provided that debt limits are never binding, such conditions reduce to

$$m_{t,t+1} u'(c_t^i) = \beta u'(c_{t+1}^i).$$

In other terms, the pricing kernel for Arrow securities coincided with the marginal rate of substitution.

To verify the second restriction ( $\ddagger$ ), consider the arbitrage given by

$$u(c_t^i - \epsilon \mathbb{E}_t m_{t,t+1} (s_{t+1}^i + f_{t+1}^i)) + \beta \mathbb{E}_t u(c_{t+1}^i + \epsilon (s_{t+1}^i + f_{t+1}^i)),$$

where  $\epsilon$  is sufficiently small. Importantly, this requires a variation of current asset holdings according to

$$s_{t+1}^i \mapsto s_{t+1}^i + \epsilon (s_{t+1}^i + f_{t+1}^i).$$

Thus,

$$s_{t+1}^i + f_{t+1}^i \geq 0$$

implies

$$s_{t+1}^i + \epsilon (s_{t+1}^i + f_{t+1}^i) + f_{t+1}^i = (1 + \epsilon) (s_{t+1}^i + f_{t+1}^i) \geq 0,$$

so ensuring that debt limits are fulfilled. As this budget-balanced readjustment of consumption cannot increase expected utility, one obtains condition ( $\ddagger$ ). To verify the first restriction ( $\dagger$ ), consider the arbitrage given by

$$u(c_t^i - \epsilon \mathbb{E}_t m_{t,t+1} h_{t+1}^i) + \beta \mathbb{E}_t u(c_{t+1}^i + \epsilon h_{t+1}^i),$$

where  $\epsilon$  is positive and sufficiently small and  $h_{t+1}^i$  is any positive contingent claim. By the usual variational argument, this yields

$$u'(c_t^i) \mathbb{E}_t m_{t,t+1} h_{t+1}^i \geq \beta \mathbb{E}_t u'(c_{t+1}^i) h_{t+1}^i,$$

which suffices to prove the validity of first-order condition ( $\dagger$ ).

A sequential equilibrium is defined as a collection of consumption plans  $(c^i)_{i \in N}$  and a sequence of prices  $(m_{t,t+1})$  such that (a) every consumption plan is optimal, for the individual, subject to sequential budget constraint, jointly with debt limits, and (b) markets clear, that is,

$$\sum_{i \in N} c_t^i = \sum_{i \in N} y_t^i = y_t \text{ and } \sum_{i \in N} s_{t+1}^i = 0.$$

Under natural debt limits (or, alternatively, a no Ponzi game condition), sequential budget constraint is equivalent to a single intertemporal budget constraint, where present value prices satisfy

$$p_{t+1} = m_{t,t+1} p_t.$$

This shows equivalence between Arrow-Debreu equilibrium and sequential equilibrium.

Consolidation of the sequence of budget constraints is as in the case of certainty. The budget constraint might be written as

$$\mathbb{E}_t p_{t+1} s_{t+1}^i + p_t c_t^i \leq p_t s_t^i + p_t y_t^i.$$

By the law of iterated expectation,

$$\mathbb{E}_0 p_{t+1} s_{t+1}^i + \mathbb{E}_0 p_t c_t^i \leq \mathbb{E}_0 p_t s_t^i + \mathbb{E}_0 p_t y_t^i.$$

Thus, consolidating over periods of trade,

$$\lim_{t \rightarrow \infty} \mathbb{E}_0 p_{t+1} s_{t+1}^i + \mathbb{E}_0 \sum_{t=0}^{\infty} p_t c_t^i \leq p_0 s_0^i + \mathbb{E}_0 \sum_{t=0}^{\infty} p_t y_t^i.$$

Furthermore, by natural debt limits,

$$\lim_{t \rightarrow \infty} \mathbb{E}_0 p_{t+1} s_{t+1}^i \geq - \lim_{t \rightarrow \infty} \mathbb{E}_0 p_{t+1} f_{t+1}^i = - \lim_{t \rightarrow \infty} \mathbb{E}_0 \sum_{k=0}^{\infty} p_{t+1+k} s_{t+1+k}^i = 0.$$

This proves that any consumption plan that is affordable under sequential budget constraint is also feasible subject to the corresponding intertemporal budget constraint. To prove the converse, it suffices to consider the financial plan given by

$$s_t^i = \frac{1}{p_t} \mathbb{E}_t \sum_{k=0}^{\infty} p_{t+k} (c_{t+k}^i - y_{t+k}^i).$$

**3.4. Implications for asset pricing.** Under complete markets, at equilibrium, prices and consumptions of individuals only depend on aggregate endowment. Thus, the stochastic discount factor is determined according to

$$m_{t,t+1} = f(y_t, y_{t+1}),$$

where  $(y_t)$  is the stochastic process for the aggregate income. Also, under constant elasticity of substitution, consumption of individuals is a constant share of aggregate income and, hence, Euler equation delivers

$$m_{t,t+1} = \beta \left( \frac{y_{t+1}}{y_t} \right)^{-\gamma}.$$

This permits to establish some relations of empirical content. For instance,

$$\sigma_t(m_{t,t+1}) \approx \gamma \sigma_t \left( \frac{y_{t+1}}{y_t} \right),$$

where  $\sigma_t$  denotes the conditional standard deviation. Hence, the volatility of the unobservable stochastic discount factor is related to the volatility of the observable rate of growth of the aggregate endowment.

## 4. NO ARBITRAGE RESTRICTIONS

**4.1. Stochastic discount factors.** Along the infinite horizon, some securities are traded, possibly contingently on revealed information. Thus, in every period of trade, conditional on acquired information, there are trades in securities under uncertainty (*ex ante*) for the delivery of claims in the following period of trade contingent on the partial resolution of uncertainty (*ex post*). The information available

at time  $t$  in  $\mathcal{T}$  is represented by an event  $E_t$  in  $\mathcal{F}_t$ ; more information will be acquired at time  $t + 1$  in  $\mathcal{T}$ , represented by the finer event  $E_{t+1} \subset E_t$  in  $\mathcal{F}_{t+1}$ .

$$\begin{array}{ccc} & \nearrow & \vdots \\ \text{information at } t & E_t & \rightarrow E_{t+1} & \text{additional information at } t + 1 \\ & \searrow & \vdots \end{array}$$

Securities are traded at event  $E_t$  in  $\mathcal{F}_t$  for the delivery of claims contingent on events  $E_{t+1} \subset E_t$  in  $\mathcal{F}_{t+1}$ . This corresponds, up to an immaterial adaptation of notation, to the description of financial markets adopted in the first part of the course (*The Fundamental Theorem of Finance*), whose analysis is directly applicable. Markets are complete if and only if, in every period of trade, any contingent claim in the following period of trade is delivered by some portfolio of available securities; markets are incomplete, otherwise.

Prices of securities exclude arbitrage across consecutive periods of trade, for otherwise such opportunities would be exploited and those prices would not persist in financial markets. By the Fundamental Theorem of Finance, the absence of arbitrage opportunities is equivalent to the existence of an adapted plan

$$(m_{t,t+1})_{t \in \mathcal{T}}$$

of strictly positive adjusted state prices, or stochastic discount factors. This plan of adjusted state prices is unique *if and only if* financial markets are complete.

An infinite-maturity security is described by its dividends,  $(y_t)_{t \in \mathcal{T}}$ , a weakly positive adapted plan. In every period of trade, an investment in this security, at market price  $q_t$ , yields a payoff  $y_{t+1} + q_{t+1}$  in the following period of trade. The payoff consists of the dividend plus the market value of the security, as the security might be dismissed. Exploiting adjusted state prices, the absence of arbitrage opportunities imposes

$$(*) \quad q_t = \mathbb{E}_t m_{t,t+1} (y_{t+1} + q_{t+1}).$$

A risk-less one-period bond is a finite maturity security yielding a constant unitary payoff only in the following period of trade. Thus, by the absence of arbitrage opportunities,

$$(**) \quad \left( \frac{1}{1 + r_t} \right) = \mathbb{E}_t m_{t,t+1},$$

where, using a canonical convention, the price of risk-less bond is identified with the reciprocal of gross interest rate.

Other finite-maturity securities might be encompassed in the analysis. Suppose that a security is issued in period  $t$  and it expires in period  $t+k+1$ . Before maturity, the security pays off dividends  $(y_{t+1+s})_{s=0}^k$ , possibly contingently on information. Prices of this security,  $(q_{t+s})_{s=0}^k$ , satisfy the following restrictions:

$$\begin{aligned} q_t &= \mathbb{E}_t m_{t,t+1} (y_{t+1} + q_{t+1}), \\ q_{t+1} &= \mathbb{E}_{t+1} m_{t+1,t+2} (y_{t+2} + q_{t+2}), \\ &\vdots \\ q_{t+k} &= \mathbb{E}_{t+k} m_{t+k,t+k+1} y_{t+k+1}. \end{aligned}$$

As a particular case, consider a two-period risk-less bond. This bond is issued at  $t$  and pays off one unit of account, uncontingently, in period  $t + 2$ . Furthermore, it can be retraded in period  $t + 1$ . No arbitrage restrictions reduce to

$$\begin{aligned} q_t &= \mathbb{E}_t m_{t,t+1} q_{t+1}, \\ q_{t+1} &= \mathbb{E}_{t+1} m_{t+1,t+2}. \end{aligned}$$

At one period from maturity, a two-period risk-less bond is priced exactly as a one period risk-less bond, as the two financial instruments are, as a matter of fact, equivalent.

Other relevant one-period securities are those delivering only contingent on some future event. Suppose that, in period  $t$ , a security is traded paying off one unit of account in period  $t + 1$  contingent on event  $E_{t+1}$  in  $\mathcal{F}_{t+1}$ . The price of this security fulfills

$$q_t = \mathbb{E}_t m_{t,t+1} \chi_{E_{t+1}},$$

where  $\chi_{E_{t+1}}$  is the indicator function for event  $E_{t+1}$  in  $\mathcal{F}_{t+1}$  (that is, given an event  $E$  in  $\mathcal{F}$ ,  $\chi_E(s) = 1$ , for  $s$  in  $E$ , and  $\chi_E(s) = 0$ , for  $s$  not in  $E$ ).

**4.2. Implicit contingent claim prices.** By concatenation of stochastic discount factors, one obtains strictly positive implicit contingent claim prices. To this purpose, let  $p_0 = 1$  and, for every  $t$  in  $\mathcal{T}$ ,

$$p_{t+1} = m_{t,t+1} p_t.$$

By construction,  $(p_t)_{t \in \mathcal{T}}$  is an adapted plan. Such prices for contingent claims are implicit as there might not be markets for these trades. Implicit contingent claim prices  $(p_t)_{t \in \mathcal{T}}$  are unique *if and only if* markets are complete.

Using implicit contingent claim prices, no arbitrage restrictions become

$$(*) \quad p_t q_t = \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}),$$

for an infinite-maturity security, and

$$(**) \quad p_t \left( \frac{1}{1 + r_t} \right) = \mathbb{E}_t p_{t+1},$$

for a one-period risk-less bond.

Consider again a finite-maturity security that is issued in period  $t$  and expires in period  $t + k + 1$ . Before maturity, this security pays off dividends  $(y_{t+1+s})_{s=0}^k$ , possibly contingently on information. Using implicit contingent claim prices, no arbitrage restrictions become

$$\begin{aligned} p_t q_t &= \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}), \\ p_{t+1} q_{t+1} &= \mathbb{E}_{t+1} p_{t+2} (y_{t+2} + q_{t+2}), \\ &\vdots \\ p_{t+k} q_{t+k} &= \mathbb{E}_{t+k} p_{t+k+1} y_{t+k+1}. \end{aligned}$$

Taking expectation, conditional on information at  $t$ , and using the law of iterated expectation, one obtains

$$\begin{aligned} p_t q_t &= \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}), \\ \mathbb{E}_t p_{t+1} q_{t+1} &= \mathbb{E}_t p_{t+2} (y_{t+2} + q_{t+2}), \\ &\vdots \\ \mathbb{E}_t p_{t+k} q_{t+k} &= \mathbb{E}_t p_{t+k+1} y_{t+k+1}. \end{aligned}$$

Adding up, and canceling out redundant terms,

$$q_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s=0}^k p_{t+1+s} y_{t+1+s}.$$

This establishes that the price of a finite-maturity security corresponds to the present value of its dividends. Notice that, even if markets are incomplete and notwithstanding the multiplicity of implicit prices of contingent claims, the present value of dividends is unambiguously defined (that is, it is the same for all implicit prices of contingent claims).

Implicit contingent claim prices can be used to price any security. However, for new securities that are not redundant (that is, such that their deliveries cannot be reproduced by portfolios of traded securities), this pricing produces ambiguous values, as markets are incomplete; that is, they only yield upper and lower bounds on values.

**4.3. Fundamental values and speculative bubbles.** It has been established that, under no arbitrage restrictions (hence, no equilibrium notion is required), a finite-maturity security is exactly priced at the present value of its dividends,

$$q_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s=0}^k p_{t+1+s} y_{t+1+s}.$$

The present value of future dividends is uniquely defined notwithstanding market incompleteness. Does an analogous relation hold for infinite-maturity securities?

An infinite-maturity security is priced according to

$$\begin{aligned} p_t q_t &= \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}), \\ p_{t+1} q_{t+1} &= \mathbb{E}_{t+1} p_{t+2} (y_{t+2} + q_{t+2}), \\ &\vdots \\ p_{t+k} q_{t+k} &= \mathbb{E}_{t+k} p_{t+k+1} (y_{t+k+1} + q_{t+k+1}). \end{aligned}$$

Taking expectation, conditional on information available at  $t$  in  $\mathcal{T}$ , and using the law of iterated expectation, one obtains

$$\begin{aligned} p_t q_t &= \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}), \\ \mathbb{E}_t p_{t+1} q_{t+1} &= \mathbb{E}_t p_{t+2} (y_{t+2} + q_{t+2}), \\ &\vdots \\ \mathbb{E}_t p_{t+k} q_{t+k} &= \mathbb{E}_t p_{t+k+1} (y_{t+k+1} + q_{t+k+1}). \end{aligned}$$

Adding up, and canceling out redundant terms,

$$q_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s=0}^k p_{t+s+1} y_{t+s+1} + \frac{1}{p_t} \mathbb{E}_t p_{t+k+1} q_{t+k+1}.$$

Going to the limit, one obtains a forward solution of the form

$$q_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s+1} y_{t+s+1} + \frac{1}{p_t} \lim_{k \in \mathcal{T}} \mathbb{E}_t p_{t+k+1} q_{t+k+1}$$

Defining

$$f_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s+1} y_{t+s+1}$$

and

$$b_t = \frac{1}{p_t} \lim_{k \in \mathcal{T}} p_{t+k+1} q_{t+k+1},$$

the price of the security decomposes as

$$(***) \quad q_t = f_t + b_t.$$

The first term is the fundamental value, the present value of dividends; the second term is the bubble component. Notice that they both depend on implicit contingent claim prices, when markets are incomplete. Furthermore, by construction, at all implicit contingent claim prices,

$$p_t f_t = \mathbb{E}_t p_{t+1} (y_{t+1} + f_{t+1})$$

and

$$p_t b_t = \mathbb{E}_t p_{t+1} b_{t+1}.$$

The latter establishes a martingale property of speculative bubbles.

Speculative bubble are the excess of the price of a security over the present value of its dividends, the fundamental value. As the determination of the present value of dividends needs implicit prices of contingent claims and these prices are not uniquely defined under incomplete markets, the fundamental value is not uniquely defined under incomplete markets, and so is the speculative bubble. The speculative bubble is *unambiguous* when it emerges at all implicit prices of contingent claims; it is *ambiguous* when it emerges only at some implicit prices of contingent claims.

## 5. EQUILIBRIUM RESTRICTIONS UNDER INCOMPLETE MARKETS

5.1. **Optimal consumption.** At prevailing prices on the market, excluding arbitrage opportunities, the individual maximizes expected intertemporal utility,

$$\mathbb{E}_0 \sum_{t \in \mathcal{T}} \beta^t u(c_t),$$

subject to budget constraints and debt limits,

$$\begin{aligned} q_t h_t + \left( \frac{1}{1+r_t} \right) b_t + (c_t - e_t) &\leq s_t, \\ (y_{t+1} + q_{t+1}) h_t + b_t &= s_{t+1}, \\ -f_{t+1} &\leq s_{t+1}, \end{aligned}$$

where initial claims are given by

$$s_0 = (y_0 + q_0) \bar{h} + \bar{b}.$$

Initial claims are inherited by the unrepresented past. Debt limits are natural, that is,

$$f_t = -\inf \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s},$$

where the infimum is taken over all adjusted state prices that are consistent with the absence of arbitrage opportunities. It can be proved that natural debt limits correspond to the maximum amounts of debt that the individual is able to honor, out of future incomes, in finite time. Such limits are clearly contingent, as contingent are future incomes.

**5.2. Euler equations.** Optimal consumption is characterized by Euler equations and transversality condition. Euler equations establish that, under budget restrictions, any slight readjustment in the portfolio of securities, inducing a substitution of consumption over consecutive periods of trade, cannot increase intertemporal expected utility. The transversality condition guarantees that any permanent contraction in the accumulation of assets, for an increase in current consumption, would require a decrease of future consumption: the individual is not over-accumulating assets. I shall focus on Euler equation and neglect, for the moment, transversality condition.

At an interior optimal consumption plan, provided that debt limits are not binding (as it is the case under Inada's condition for natural debt limits), Euler equations impose

$$(*) \quad u'(c_t) q_t = \beta \mathbb{E}_t u'(c_{t+1}) (q_{t+1} + y_{t+1})$$

and

$$(**) \quad u'(c_t) \left( \frac{1}{1+r_t} \right) = \beta \mathbb{E}_t u'(c_{t+1}).$$

These are necessary conditions for optimality, which are obtained by elementary arbitrage arguments: any small readjustment in the holding of securities cannot increase expected utility over consecutive periods of trade.

The form of Euler equations should not be surprising: these restrictions simply establish that marginal rates of substitution correspond to some adapted plan of stochastic discount factors,

$$m_{t,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

This happens also under complete markets. The crucial difference is that, when markets are complete, the stochastic discount factor is unique and Euler equations establish coincidence between this unique stochastic discount factor and the marginal rate of substitution. Differently, when markets are incomplete, there exists a multiplicity of stochastic discount factors and Euler equations simply impose coincidence between one of such stochastic discount factors and the marginal rate of substitution. The relevant implication is that different individuals, under incomplete markets, might have different marginal rates of substitution.

For iso-elastic per-period utilities, first-order conditions take the form

$$(*) \quad q_t = \beta \mathbb{E}_t \left( \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}} (q_{t+1} + y_{t+1})$$

and

$$(**) \quad \left( \frac{1}{1+r_t} \right) = \beta \mathbb{E}_t \left( \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}},$$

where  $\sigma > 0$  is the constant elasticity of intertemporal substitution. Thus, the stochastic discount factor obeys the rule

$$m_{t,t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\sigma}}.$$

This particular form of stochastic discount factor is typically exploited in empirical studies.

**5.3. Equilibrium.** Competitive equilibrium is defined as for sequential trades under complete markets. The economy consists of different individuals, a finite set  $\mathcal{J}$ . Such individuals might differ in preferences over consumption plans and incomes. Given prices of the infinite-maturity security and interest rates, each individual formulates optimal consumption and investment plans subject to budget restrictions. At equilibrium, markets for consumption claims and securities clear.

Formally, an equilibrium is defined by prices of the infinite-maturity security, interest rates and plans of individuals,

$$\left( (q_t)_{t \in \mathcal{T}}, (r_t)_{t \in \mathcal{T}}, \left( (c_t^i, b_t^i, h_t^i)_{t \in \mathcal{T}} \right)_{i \in \mathcal{J}} \right),$$

satisfying the following restrictions:

- (a) Plans of individuals are optimal subject to budget restrictions, that is, for every individual  $i$  in  $\mathcal{J}$ , the consumption plan  $(c_t^i)_{t \in \mathcal{T}}$ , maximizes expected utility,

$$\mathbb{E}_0 \sum_{t \in \mathcal{T}} \beta^t u^i(c_t^i),$$

subject to budget constraints and debt limits, at every  $t$  in  $\mathcal{T}$ ,

$$\begin{aligned} q_t h_t^i + \left( \frac{1}{1+r_t} \right) b_t^i + (c_t^i - e_t^i) &\leq s_t^i, \\ (y_{t+1} + q_{t+1}) h_t^i + b_t^i &= s_{t+1}^i, \\ - \inf \frac{1}{p_{t+1}} \mathbb{E}_{t+1} \sum_{s \in \mathcal{T}} p_{t+1+s} e_{t+1+s}^i &\leq s_{t+1}^i, \end{aligned}$$

where initial claims are given by

$$s_0^i = (y_0 + q_0) \bar{h}^i + \bar{b}^i.$$

Across individuals, initial holdings of assets satisfy

$$\bar{h} = \sum_{i \in \mathcal{J}} \bar{h}^i = 1 \text{ and } \bar{b} = \sum_{i \in \mathcal{J}} \bar{b}^i = 0.$$

Without loss of generality, the aggregate supply of the infinite-maturity security is unitary.

- (c) Markets for consumption claims, infinite-maturity security and bonds clear, that is, at every  $t$  in  $\mathcal{T}$ ,

$$\begin{aligned}\sum_{i \in \mathcal{J}} c_t^i &= \sum_{i \in \mathcal{J}} e_t^i + y_t, \\ \sum_{i \in \mathcal{J}} h_t^i &= 1, \\ \sum_{i \in \mathcal{J}} b_t^i &= 0.\end{aligned}$$

At equilibrium, because of market incompleteness, marginal rates of substitution might not be equalized across individuals. That is, it might happen (and typically happens) that, for two distinct individuals  $(i_1, i_2)$  in  $\mathcal{J} \times \mathcal{J}$ , at equilibrium,

$$m_{t,t+1}^{i_1} = \frac{\beta u'(c_{t+1}^{i_1})}{u'(c_t^{i_1})} \neq \frac{\beta u'(c_{t+1}^{i_2})}{u'(c_t^{i_2})} = m_{t,t+1}^{i_2}.$$

Indeed, such marginal evaluations coincide only over trades that are feasible given securities available on the market (Euler equations hold true for all individuals). As disparities in subjective evaluations of risk persist, equilibrium is not Pareto efficient: an hypothetical planner might improve social welfare by redistributing from individuals with low evaluation to individuals with high evaluation. Because of this inefficiency, the determination of equilibrium prices is not as easy as in the case of complete markets.

**5.4. Representative individual hypothesis.** The *representative individual hypothesis* simplifies the determination of equilibrium prices. Under the assumption of a single individual (or of many identical individuals), market clearing restrictions are trivial: optimal consumption, at equilibrium, corresponds to available resources in the aggregate,

$$c_t = e_t + y_t.$$

Yet, though market clearing is trivial, optimality of consumption yields restrictions of empirical content: prices need to adjust so as to induce the representative individual to exactly consume available resources in the aggregate. To summarize, under the representative individual hypothesis, the only relevant equilibrium restrictions are given by Euler equations (plus the neglected transversality condition) evaluated at the autarchic consumption (the representative individual consumes the endowment and dividends from the ownership of the infinite-maturity security). Individual consumption, under the representative individual hypothesis, corresponds to aggregate consumption.

## 6. EMPIRICAL CONTENT

**6.1. Risk-aversion corrections.** Define risky return and risk-less return, respectively, as

$$R_{t,t+1}^s = \frac{q_{t+1} + y_{t+1}}{q_t}$$

and

$$R_{t,t+1}^f = 1 + r_t.$$

Notice that the risk-less return is only contingent on information at  $t$ , whereas the risky return depends on information at  $t + 1$ . Arbitrage restrictions (\*)-(\*\*) reduce to

$$1 = \mathbb{E}_t m_{t,t+1} R_{t,t+1}^s$$

and

$$1 = R_{t,t+1}^f \mathbb{E}_t m_{t,t+1}.$$

These two restrictions yield

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = -R_{t,t+1}^f \text{cov}_t(m_{t,t+1}, R_{t,t+1}^s).$$

An investment in a risky security is remunerated at the risk-less return plus a compensation (*risk premium*) for risk. The risk premium (the excess of expected risky return over the risk-less return) is negatively proportional to the covariance between the stochastic discount factor and the risky return. Furthermore, it is the correlation between the stochastic discount factor and the risky return that generates the premium, and not simply the risk of the investment.

To understand the role of risk aversion in the determination of the risk premium, remember that, by Euler equations (\*)-(\*\*), the marginal rate of substitution of an investor coincides with a stochastic discount factor,

$$m_{t,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

Thus, the risk premium is determined by

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = -\frac{\text{cov}_t(u'(c_{t+1}), R_{t,t+1}^s)}{\mathbb{E}_t u'(c_{t+1})}.$$

Under risk-neutrality, marginal utility is constant, so that no compensation for risk is needed,

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = -\frac{\text{cov}_t(u'(c_{t+1}), R_{t,t+1}^s)}{\mathbb{E}_t u'(c_{t+1})} = 0.$$

The expected risky return is equal to the risk-less return. Under risk-aversion, an investment in a risky security need be remunerated at risk-less return plus a risk premium, as a risk-less investment opportunity is available. This being true, can the risk premium be negative? Why is a risk-averse investor willing to accept a negative risk premium over the risky security?

There is nothing puzzling in a negative risk premium. Marginal utility is a measure of the benefit from additional consumption. When the risky return is positively correlated with marginal utility, the security yields, on average, high return, when consumption is marginally more valuable, and low return, when consumption is marginally less valuable. This benefit is compensated by a negative risk-premium, as the security is useful for insurance even though its expected return falls below the risk-less return. When the risky return is negatively correlated with marginal utility, the security yields, on average, high return, when consumption is marginally less valuable, and low return, when consumption is marginally more valuable. This security produces a benefit only because its expected return is above the risk-less return.

**6.2. Martingale theory of stock prices.** Using arbitrage-free restrictions (\*)-(\*\*),

$$q_t = \left( \frac{1}{1 + r_t} \right) \mathbb{E}_t (y_{t+1} + q_{t+1}) + \text{cov}_t (m_{t,t+1}, y_{t+1} + q_{t+1}).$$

This is the central equation in modern finance. The price of a security decomposes into two terms: the first term is the risk-neutral present value of payoffs; the second term is the crucial discount for risk; a large negative covariance implies a low price for the security.

From an investor's Euler equations (\*)-(\*\*), the stochastic discount factor corresponds to the marginal rate of substitution,

$$m_{t,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

By decreasing marginal utility, this relation establishes that the stochastic discount factor is relatively high, when consumption decreases, and relatively low, when consumption increases. A positive covariance basically means that the payoff of the security is, on average, high (low) when consumption is low (high) and, hence, marginally more (less) valuable for an investor. As this security yields high (low) payoff when payoff is more (less) desirable, the price is driven up on the market.

When the covariance term vanishes, the price of an infinite-maturity security obey the rule

$$q_t = \left( \frac{1}{1 + r_t} \right) \mathbb{E}_t (y_{t+1} + q_{t+1}).$$

In turn, the covariance is null when marginal utility is constant (that is, under risk-neutrality) or, alternatively, when consumption is constant. The above relation establishes that the security is simply priced at the expected payoff, discounted by risk-less gross interest rate.

When the security pays no dividends and interest rate vanishes (these assumptions might be justified over short-horizons), the equation reduces to

$$q_t = \mathbb{E}_t q_{t+1}.$$

Equivalently, security prices follow a time-series process of the form

$$q_{t+1} = q_t + \epsilon_{t+1},$$

where  $\epsilon_{t+1}$  is a random innovation satisfying  $\mathbb{E}_t \epsilon_{t+1} = 0$ . If the variance of the innovation is constant, prices follow a *random walk*. More generally, prices follow a *martingale*. Accordingly, returns on securities should not be predictable, as unpredictable are coin flips. This is the content of the *efficient-market hypothesis*: financial markets are informationally efficient, that is, prices on traded assets (*e.g.*, stocks, bonds, or property) already reflect all known information. The efficient-market hypothesis states that it is impossible to consistently outperform the market by using any information already available in the market.

**6.3. Capital asset pricing model.** Euler equations (\*)-(\*\*) imply

$$\mathbb{E}_t \left( R_{t,t+1}^s - R_{t,t+1}^f \right) u'(c_{t+1}) = 0.$$

Assuming quadratic utility,  $u'(c) = \bar{c} - c$ , where  $\bar{c}$  is a sufficiently large satiation consumption level. Thus,

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = R_{t,t+1}^f \frac{\text{cov}_t(c_{t+1}, R_{t,t+1}^s)}{\mathbb{E}_t u'(c_{t+1})}.$$

The risk premium is proportional to the covariance between risky return and consumption. A quadratic utility might be justified as an approximation of utility up to the second-order term. Under quadratic utility, only the first and second moment of the distribution (mean and variance) are relevant for expected utility.

Assume now that there exists a risky security whose return is perfectly correlated with consumption, that is, for instance,  $R_{t,t+1}^m = c_{t+1}$ . This is a sort of aggregate, or market, security. The above relation yields

$$\begin{aligned} \mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f &= \frac{\text{cov}_t(R_{t,t+1}^m, R_{t,t+1}^s)}{\text{var}_t(R_{t,t+1}^m)} \left( \mathbb{E}_t R_{t,t+1}^m - R_{t,t+1}^f \right) \\ &= \beta_t^s \left( \mathbb{E}_t R_{t,t+1}^m - R_{t,t+1}^f \right). \end{aligned}$$

This is the fundamental relation of the *capital asset pricing model (CAPM)*. It asserts that the risk-premium of an investment is proportional to the market risk-premium (or the price of risk).

**6.4. Equity premium puzzle.** Moving from the established relation

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = -R_{t,t+1}^f \text{cov}_t(m_{t,t+1}, R_{t,t+1}^s),$$

one obtains

$$\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f = -R_{t,t+1}^f \rho_t(m_{t,t+1}, R_{t,t+1}^s) \sigma_t(m_{t,t+1}) \sigma_t(R_{t,t+1}^s),$$

where

$$\rho_t(m_{t,t+1}, R_{t,t+1}^s) = \frac{\text{cov}_t(m_{t,t+1}, R_{t,t+1}^s)}{\sigma_t(m_{t,t+1}) \sigma_t(R_{t,t+1}^s)}$$

denotes the correlation between the stochastic discount factor and the risky return, with  $\sigma_t(m_{t,t+1})$  and  $\sigma_t(R_{t,t+1}^s)$  being standard deviations. As

$$|\rho_t(m_{t,t+1}, R_{t,t+1}^s)| \leq 1,$$

one obtains

$$\sigma_t(m_{t,t+1}) \geq \frac{|\mathbb{E}_t R_{t,t+1}^s - R_{t,t+1}^f|}{R_{t,t+1}^f \sigma_t(R_{t,t+1}^s)}.$$

The right hand-side is the ratio of the excess return (or risk-premium) into the standard deviation of a given return. This is known as *Sharpe ratio*. Thus, the volatility of stochastic discount factor (and, hence, of marginal utilities) is restricted by Sharpe ratio on traded stocks. This is of empirical content. Indeed, Sharpe ratios are observable and these observations impose a limit on the variance of unobservable stochastic discount factors.

The bounds provided by Sharpe ratios are puzzling when confronted with aggregate stock market data. On the one side, it is observed that the risk premium on some stock market indexes is high relatively to the volatility of the returns of these indexes. Consistently, Sharpe ratios on these indexes are high and bounds on the volatility of marginal utilities are high. On the other hand, it is observed a low volatility of consumption. These two facts can be reconciled with the predictions of

the theory only under an extreme risk aversion, that is, a large variation in marginal utility induced by a small variation in consumption. This seems to contradict the empirical evidence on risk aversion. This anomaly is commonly referred to as the *equity premium puzzle*.

## 7. INTERTEMPORAL BUDGET CONSTRAINT AND OPTIMALITY

**7.1. Consolidation of budget constraints.** When markets are incomplete, there exist a multiplicity of implicit contingent claim prices under no arbitrage restrictions. Peg any of such processes of contingent claim prices inducing a finite present value of future endowment, that is, such that, at every  $t$  in  $\mathcal{T}$ ,

$$\frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s} \text{ is finite.}$$

I here show that any sequentially budget-feasible consumption plan satisfies the intertemporal budget constraint evaluated at those implicit contingent claim prices, that is, at every  $t$  in  $\mathcal{T}$ ,

$$(\dagger) \quad \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} c_{t+s} \leq p_t s_t + \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s}.$$

Given prices of contingent claims, using no arbitrage restrictions (\*)-(\*\*),

$$\mathbb{E}_t p_{t+1} s_{t+1} = \mathbb{E}_t p_{t+1} (y_{t+1} + q_{t+1}) h_t + \mathbb{E}_t p_{t+1} b_t = p_t q_t h_t + p_t \left( \frac{1}{1+r_t} \right) b_t.$$

Hence, the budget constraint implies

$$\begin{aligned} \mathbb{E}_t p_{t+1} s_{t+1} + p_t (c_t - e_t) &\leq p_t s_t, \\ \mathbb{E}_{t+1} p_{t+2} s_{t+2} + p_{t+1} (c_{t+1} - e_{t+1}) &\leq p_{t+1} s_{t+1}, \\ &\vdots \\ \mathbb{E}_{t+k} p_{t+k+1} s_{t+k+1} + p_{t+k} (c_{t+k} - e_{t+k}) &\leq p_{t+k} s_{t+k}. \end{aligned}$$

Taking the expectation, conditional on information available at  $t$  in  $\mathcal{T}$ , and using the law of iterated expectation,

$$\begin{aligned} \mathbb{E}_t p_{t+1} s_{t+1} + p_t (c_t - e_t) &\leq p_t s_t, \\ \mathbb{E}_t p_{t+2} s_{t+2} + \mathbb{E}_t p_{t+1} (c_{t+1} - e_{t+1}) &\leq \mathbb{E}_t p_{t+1} s_{t+1}, \\ &\vdots \\ \mathbb{E}_t p_{t+k+1} s_{t+k+1} + \mathbb{E}_t p_{t+k} (c_{t+k} - e_{t+k}) &\leq \mathbb{E}_t p_{t+k} s_{t+k}. \end{aligned}$$

By addition, eliminating redundant terms,

$$\mathbb{E}_t p_{t+k+1} s_{t+k+1} + \mathbb{E}_t \sum_{s=0}^k p_{t+s} c_{t+s} \leq p_t s_t + \mathbb{E}_t \sum_{s=0}^k p_{t+s} e_{t+s}.$$

Furthermore, by natural debt limits,

$$\mathbb{E}_t p_{t+k+1} s_{t+k+1} \geq -\mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+k+s+1} e_{t+k+s+1}.$$

It follows that

$$\mathbb{E}_t \sum_{s=0}^k p_{t+s} c_{t+s} \leq p_t s_t + \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s}.$$

As this holds true for every arbitrary  $k$  in  $\mathcal{T}$ , going to the limit, restriction (†) holds true. Therefore, a sequentially budget-feasible consumption plan satisfies the intertemporal budget constraint evaluated at implicit prices for contingent claims giving a finite value to future endowments.

**7.2. Transversality condition under uniform impatience.** I here show that, under the additional hypothesis of uniform impatience, at an interior optimal consumption plan, the intertemporal budget constraint is exhausted, when evaluated at any process of contingent claim prices inducing a finite present value of future endowment, that is, such that, at every  $t$  in  $\mathcal{T}$ ,

$$\frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s} \text{ is finite.}$$

Because condition (†) holds true, it suffices to prove that, at every  $t$  in  $\mathcal{T}$ ,

$$(\ddagger) \quad \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} c_{t+s} \geq p_t s_t + \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s}.$$

The exhaustion of intertemporal budget is a strong form of *transversality condition*.

Assume that the individual is *uniformly impatient*, that is, there exists  $1 > \eta > 0$  such that, at every interior consumption plan, in every period of trade  $t$  in  $\mathcal{T}$ ,

$$(\S) \quad \beta^t u'(c_t) - \eta \mathbb{E}_t \sum_{s \in \mathcal{T}} \beta^{t+s+1} u'(c_{t+s+1}) c_{t+s+1} \geq 0.$$

This establishes that, in terms of first-order effects, any unitary expansion of current consumption, against a permanent contraction of future consumption of an amount  $\eta > 0$ , increases expected intertemporal utility. In other terms, the subjective rate of substitution of permanent future consumption for current consumption does not fall below some given  $\eta > 0$ .

The intuition for the transversality condition is very simple. Under uniform impatience, at an interior optimal consumption plan, the value of asset accumulation is uniformly bounded by the reciprocal of  $\eta > 0$ . Indeed, supposing not, at some period of trade, contingent on some event, the individual could permanently decrease asset accumulation of a proportion  $\eta > 0$ , along with future consumption. This permanent disinvestment would allow for an unitary increase in current consumption. By uniform impatience (§), such a reallocation of resources would yield higher intertemporal expected utility, contradicting optimality.

I shall now show that, by the hypothesis of uniform impatience, at any optimal interior consumption plan,

$$\eta \left( \left( \frac{1}{1+r_t} \right) b_t + q_t h_t \right) \leq 1$$

or, equivalently, by no arbitrage restrictions (\*)-(\*\*), at all processes of implicit contingent claim prices,

$$(\diamond) \quad \eta \mathbb{E}_t p_{t+1} s_{t+1} \leq p_t.$$

Indeed, given  $t$  in  $\mathcal{T}$ , consider the alternative consumption and investment plans given, at every  $k$  in  $\mathcal{T}$ , by:

$$\begin{aligned}\bar{c}_t &= c_t + \eta\xi \frac{1}{p_t} \mathbb{E}_t p_{t+1} s_{t+1}, \\ \bar{c}_{t+k+1} &= (1 - \eta\xi) c_{t+k+1}, \\ \bar{b}_{t+k} &= (1 - \eta\xi) b_{t+k}, \\ \bar{h}_{t+k} &= (1 - \eta\xi) h_{t+k},\end{aligned}$$

where  $\xi > 0$  is sufficiently small. Consumption and investment plans are not modified in the previous periods. It is easily verified that these adjusted consumption and investment plans satisfy the sequence of budget constraints and debt limit constraints. Indeed, in period  $t$  in  $\mathcal{T}$ , resources, made available from the permanent contraction of investments in assets, are exploited to increase current consumption; in the following periods, the permanent contraction in the holding of securities is budget-compensated by a contraction in consumption. As this cannot produce a welfare-improvement at optimal consumption,

$$\frac{1}{\xi} \left( \mathbb{E}_t \sum_{s \in \mathcal{T}} \beta^{t+s} u(\bar{c}_{t+s}) - \mathbb{E}_t \sum_{s \in \mathcal{T}} \beta^{t+s} u(c_{t+s}) \right) \leq 0.$$

Letting  $\xi > 0$  vanish, this yields a negative first-order effect,

$$\beta^t u'(c_t) \left( \eta \frac{1}{p_t} \mathbb{E}_t p_{t+1} s_{t+1} \right) - \eta \mathbb{E}_t \sum_{s \in \mathcal{T}} \beta^{t+s+1} u'(c_{t+1+s}) c_{t+1+s} \leq 0.$$

Hence, because of uniform impatience (§), restriction (◇) must hold true at every period  $t$  in  $\mathcal{T}$ .

Consolidating the sequence of budget constraints, observing that budget constraints hold with equality at an optimal consumption plan and using condition (◇),

$$\begin{aligned}\frac{1}{\eta} \mathbb{E}_t p_{t+k} + \mathbb{E}_t \sum_{s=0}^k p_{t+s} c_{t+s} &\geq \\ \mathbb{E}_t p_{t+k+1} s_{t+k+1} + \mathbb{E}_t \sum_{s=0}^k p_{t+s} c_{t+s} &= p_t s_t + \mathbb{E}_t \sum_{s=0}^k p_{t+s} e_{t+s}.\end{aligned}$$

Going to the limit,

$$\lim_{k \in \mathcal{T}} \frac{1}{\eta} \mathbb{E}_t p_{t+k} + \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} c_{t+s} \geq p_t s_t + \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s}.$$

Notice that all series must be finite (in the right-hand side, by assumption; in the left hand-side, because of (†)). Finally, observe that, by interiority of the optimal consumption plan, there exists  $\epsilon > 0$  such that  $c_{t+s} \geq \epsilon$ , for every  $s$  in  $\mathcal{T}$ , and, hence,

$$\epsilon \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} \leq \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} c_{t+s}.$$

This definitely implies that

$$\lim_{k \in \mathcal{T}} \frac{1}{\eta} \mathbb{E}_t p_{t+k} = 0.$$

Hence, condition (‡) is satisfied. At an optimal interior consumption plan, under the hypothesis of uniform impatience, the intertemporal budget constraint is exhausted for all processes of implicit prices of contingent claims, provided that the present value of endowment is finite. This is a sort of strong form of *transversality condition*.

## 8. RULING OUT SPECULATIVE BUBBLES AT EQUILIBRIUM

Speculative bubbles are perfectly consistent with no arbitrage restrictions. Are they ruled out by equilibrium restrictions? Ambiguous speculative bubbles cannot occur at an interior equilibrium under additional restrictions. First, the infinite-maturity security must be sufficiently productive, that is, for some sufficiently large  $\lambda > 0$ ,

$$e_t \leq \lambda y_t.$$

This establishes that the (possibly stochastic) rate of growth of dividends does not fall below the (possibly stochastic) rate of growth of aggregate non-financial income. Second, individuals are uniformly impatient, that is, restriction (§) is satisfied.

Notice that, by arbitrage restrictions, for all processes of implicit contingent claim prices,

$$q_t = f_t + b_t \geq f_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s+1} y_{t+s+1}.$$

Hence, by the hypothesis of a sufficiently productive infinite-maturity security, for all processes of implicit prices of contingent claims, the present value of future incomes is finite, that is,

$$\frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} e_{t+s} \text{ is finite.}$$

Notice that this might not be true at equilibrium, as, in order to natural debt limits not to be vacuous and optimal consumption plans to exist, only the worst evaluation of the present value of future incomes needs be finite.

Peg any process of implicit prices of contingent claims. By uniform impatience, the intertemporal budget constraint is exhausted (†)-(‡), that is, at every  $t$  in  $\mathcal{T}$ ,

$$\frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s} (c_{t+s} - e_{t+s}) = s_t.$$

By market clearing for consumption ( $c_{t+s} = e_{t+s} + y_{t+s}$  for every  $s$  in  $\mathcal{T}$ ),

$$\frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s+1} y_{t+s+1} = s_t - (c_t - e_t).$$

By market clearing for the infinite-maturity security and the risk-less bond, using the budget constraint,

$$q_t = \left( \frac{1}{1 + r_t} \right) b_t + q_t h_t = s_t - (c_t - e_t).$$

Therefore,

$$f_t = \frac{1}{p_t} \mathbb{E}_t \sum_{s \in \mathcal{T}} p_{t+s+1} y_{t+s+1} = q_t,$$

so implying that the speculative component is null.

Though I have maintained the hypothesis of a representative individual to simplify the presentation, the argument straightforwardly extends to heterogeneous individuals. It simply exploits a sort of intertemporal Walras' Law.