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THE LAWS OF NATURE AND THE EFFECTIVENESS OF MATHEMATICS

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Abstract: In this paper I try to evaluate what I regard as the main attempts at explaining the effectiveness of mathematics in the natural sciences, namely (1) Antinaturalism, (2) Kantism, (3) Semanticism, (4) Algorithmic Complexity Theory. The first position has been defended by Mark Steiner, who claims that the “user friendliness” of nature for the applied mathematician is the best argument against a naturalistic explanation of the origin of the universe. The second is naturalistic and mixes the Kantian tradition with evolutionary studies about our innate mathematical abilities. The third turns to the Fregean tradition and considers mathematics a particular kind of language, thus treating the effectiveness of mathematics as a particular instance of the effectiveness of natural languages. The fourth hypothesis, building on formal results by Kolmogorov, Solomonov and Chaitin, claims that mathematics is so useful in describing the natural world because it is the science of the abbreviation of sequences, and mathematically formulated laws of nature enable us to compress the information contained in the sequence of numbers in which we code our observations. In this tradition, laws are equivalent to the shortest algorithms capable of generating the lists of zeros and ones representing the empirical data. Along the way, I present and reject the “deflationary explanation”, which claims that in wondering about the applicability of so many mathematical structures to nature, we tend to forget the many cases in which no application is possible.

Key words: mathematics; laws of nature; algorithmic complexity theory; evolution; semantics.

Our experience hitherto justifies us in believing that nature is the
realization of the simplest conceivable mathematical ideas.
(Einstein, 1933)

1. INTRODUCTION

In this note I will try to connect the difficult question of the effectiveness of mathematics in the natural science (Wigner, 1967) with the philosophical issue of the nature of natural laws. Trying to create a bridge between these two as-yet unrelated areas of philosophical research seems a fruitful enterprise as it could shed light on both. On the one hand, for example, the philosophical questions of (i) what laws *are* (ontological and semantic realism vs. ontological and semantic antirealism about laws) and (ii) how we come to know them, seem to be questions that can be fruitfully approached anew if we pay attention to the mathematical character of laws: the idealized, abstract and simplified character of laws in our models is often due to the need of having *tractable* mathematical problems and solutions. This is, for instance, why *stable* solutions to *linear* differential equations were privileged at the beginning of modern mathematical physics. On the other hand, the “unreasonable effectiveness of mathematics” is an abbreviated slogan to refer to the *descriptive, predictive and explanatory power of mathematics in dealing with the natural (and social) world*, a power that almost exclusively depends on the fact that laws in most natural and social sciences are expressed in mathematical language. The fact that scientific descriptions, predictions and explanations are often a matter of *subsuming single phenomena under laws* (together, of course, with a specification of boundary conditions and/or initial conditions), confirms the importance of establishing a relationship between the problem of the effectiveness of mathematics and the philosophy of the laws of nature.

As a further illustration of the opportunity of a cross-fertilization of the two areas above, consider that the ontological realist about laws believes that there are mind-independent truth-makers (for instance, relations about properties of physical systems) making any well-confirmed law-statement true. What is the character of such truth-makers? It is an interesting and not-yet sufficiently explored question whether the entities referred to by such statements are describable in a sufficiently faithful way in a non-mathematical language. If this were not feasible, we might have to conclude in a Pythagorean fashion that the ultimate structure of the universe is mathematical in character. As Steiner put it (Steiner, 1998):

Can we specify, by using a non-mathematical language, how can the world be made in such a way that valid mathematical deductions are effective in predicting observations? (p. 24)

I should add that rather than offering alleged “definitive solutions” to the hard problems raised above, the paper tries to suggest new directions of inquiry by reviewing and briefly evaluating the scanty available literature. In the next section, I will try to provide arguments to the effect that the problem of the effectiveness of mathematics is a *genuine* one. Such a preliminary task is necessary, since if the problem posed by Wigner were a pseudo-problem, the claim that it could help us to look at the issue of scientific laws under a new light would be groundless. In the third section, I will present and briefly evaluate the current attempts at answering Wigner’s problem (I will list four of them). In the fourth section, I will focus on one of these attempts, centred on the view that laws of nature are the *software* of the physical universe (algorithmic view of the laws of nature and of the applicability of mathematics). In this metaphor, which for its proponents is suggestive of a deeper truth about mathematically formulated laws, the universe is considered to be a gigantic computer whose *hardware* is whatever fundamental physics tells us about the ultimate component of matter (fields, particles, superstrings, etc) and whose software abbreviates and compresses the ordered sequence of states it goes through in time. In the fifth section, I will raise various difficulties to the algorithmic conception of laws, some of which may appear fatal to the whole project.

2. SOME SCEPTICAL REMARKS AGAINST WIGNER’S QUESTION

As is often the case in philosophy, one often wonders whether an apparently deep problem to a closer analysis might reveal itself but a pseudo problem. In our case, a dissolution of the question why mathematics can be used to describe, predict and even explain physical phenomena – think of the structural-geometrical *explanation* of the time-dilation or of the length-contraction effects in terms of the *geometry* of Minkowski space-time, in which an invariant *four-dimensional* entity is projected onto different relative 3-spaces with its own inertial time – is the remark that *the effectiveness of mathematics meant in this sense results from a selective effect, making us focus only on the evidence of success*. Ignoring evidence of failure, one can make any hypothesis look good, so that one could as well refer to the un-effectiveness (or the very frequent failure) of many mathematical theories to apply to the natural world. A second objection consists in the fact that very often, as in the case of calculus at the times of

Newton and Leibniz, pieces of mathematics are invented and constructed with the intention to solve physical problems: no wonder that they sometimes work!

In order to tackle the first objection, note that we cannot take statistics in order to determine the fraction of areas of mathematics that have had a successful application over the total number of areas of mathematical research. Therefore, since the objection presupposes such a statistical count, it loses its force. However, even if it did make sense to take such a counting, we would not find many branches of pure, non-applied mathematics that are not somehow connected with the empirical world.¹

Of course, we must agree that we can apply mathematics to the empirical reality only in some few, lucky cases and not in others, namely when we are dealing with simplified or “simple” enough phenomena². Cases in which the mathematician cannot find the way to describe a natural process clearly outnumber those in which she can (see Steiner, 1998, p. 9).

However, I take it that it is still mysterious that *some* of the consequences of our mathematical symbols are also consequences of the symbolized phenomena, as Hertz once put it, especially when the symbols were not created for applicative purposes. If the application of a part of mathematics to physics has been, at least sometimes, unpredictable and unexpected – that is, if that part of mathematics had been constructed for pure, or non-applicative purposes – *then I claim that even a single instance of successful applicability would cry for an explanation.*

This remark takes us to the second reply to our question: for instance there are parts of mathematics like group theory that had not been invented to deal with physical problems, but nowadays it would be difficult to deal with the zoo of elementary particles without using the algebra of groups. The use of Levi Civita’s and Ricci’s absolute differential calculus in the general theory of relativity is an often quoted instance of a surprising spin off that had not been pursued intentionally. Dirac discovered his equation (and the existence of antimatter) by working on a formal, syntactical analogy with what Pauli had done before him with 2x2 matrices: despite the fact that Dirac was trying to solve a physical problem, it is still mysterious that merely formal analogies like these can sometimes be conducive to truth or success. In his characteristically direct and non-pompous style, Feynman put it thus (Feynman, 1967):

¹ By “connected” I mean to refer also to cases like the application of number theory to the problem of secret coding.

² Simple here is almost a synonymous of “can be captured by a mathematical model”.

I find it quite amazing that it is possible to predict what will happen by mathematics, which is simply following rules which really have nothing to do with the original thing. (p. 171)

Finally, we should also note that an explanation for the applicative success of mathematics cannot be easily found just by taking a philosophical stance on the ontology of mathematical objects. In fact, just to name two positions, the problem of the applicability of mathematics creates troubles for both the constructivists and the platonists. The former must explain why a *creation* of ours, often pursued for subjective purposes and without any pragmatist interest, can carry so much descriptive and predictive power, enabling us to explain and classify entities of the natural world that we did *not* create (under the typical assumption of the scientific realist). The platonists should explain why the natural world should conform or be partially isomorphic to the abstract realm populated by mathematical facts and entities, besides answering the usual objection coming from the causal theory of knowledge. If the abstract real is causally inert, how can we get to know it?

3. FOUR POSITIONS ON WIGNER'S QUESTION

3.1 Steiner's antinaturalism

The first position I will list, which is *antinaturalistic*, has been explicitly defended by Steiner (1998): the universe is “user friendly” for the mathematician, but no explanation in natural terms seems available. Here, anthropomorphism strikes back, with a theological undertone (Steiner, 1998, p.10). According to Steiner (1998):

to use mathematics to define similarity and analogy in physics is almost as anthropocentric as using “male-female” or “earthly-heavenly” as classifying tools. Why? Because the concept of mathematics itself is species-specific. There is no objective criterion for a structure to be mathematics – and not every structure count as a mathematical structure...Mathematicians today have adopted internal criteria to decide whether to study a structure as mathematical. Two of these are *beauty* and *convenience*. (pp. 8-9).

The interest of Steiner's book lies more in providing interesting historical examples of unexpected applications of mathematical concepts to the physical world via mathematical, formal analogies than in trying to explain the problem at stake. Concluding that the effectiveness of mathematics

amounts to an anthropomorphic (“childish-like”) projection of our constructed structures onto the physical world is simply a re-description of what needs to be explained, namely that mathematics, viewed as a social construction, somehow miraculously captures parts of a universe that is independent of our minds.

Another “risk” of Steiner’s antinaturalistic position is that of encouraging a sort of mysticism about Wigner’s problem: if naturalism means (i) that the existing or the real coincides with what exists in space and time, (ii) that we have the power in principle to investigate and get to know reality *via* the discovery of natural laws, (iii) that we can also find out how we get to know the world, antinaturalism might mean that we will never find out through science why our mathematically formulated laws enable us to predict and classify natural entities. This would somehow mean the bankruptcy of the possibility of coming up with an explanation to our question. We can therefore move on to a different attempt at explaining the problem.

3.2 The Kantian answer

The second position could be called a form of “kantism”: mathematics works because, roughly speaking, we perceive the world by using inner mathematical intuitions (space and time), that is, we project *a priori* forms, constructions and categories of our own onto our experience of nature. Here the explanation of the effectiveness of mathematics can rely on data coming from evolutionary and cognitive psychology and as such it is not at all antinaturalistic (Longo, 2000a, 2000b). On the contrary, its promising aspect is the fact that it can be measured against in principle available *empirical evidence*, like data on arithmetical abilities of other animals, based on the evolutionary advantage of being able to distinguish, for example, “one predator” from “many predators” or even “one” from “two” or “three” (Devlin, 1999)

Just to offer an argument based on mere *a priori* plausibility of a Kant-type explanation, one could reason as follows. Since *mathematical abilities* are to a good extent genetically determined, fundamental mathematical concepts, like *number* or *space*, might be *a priori*, in the same sense in which fundamental concepts are *a priori* in Fodor’s *language of thought*. Otherwise, one could ask, what would be the *object* of such mathematical abilities? Furthermore, *if* the contents of our thoughts are expressed in symbolic structures of an innate language, whose syntax and semantics are similar to (though more abstract than) those of the natural language, *then* the claim that all our mental processes are essentially *computational* could explain why the development and justification – not thought necessarily the

origin of certain concepts – of mathematical knowledge is *a priori* without invoking any outlandish form of Platonism.

Note that the question of applicability within this second attempt would be solved only in part, because while it can be true that we perceive the world also thanks to “primitive mathematical intuitions”, it remains to be explained why developments of mathematics that are very remote from those intuitions can still have an empirical application. The question of the nature of laws of nature is therefore not even touched by this second attempt.

One should in fact consider that mathematics is applicable also at scales (like those typical of atoms and subatomic particles on the one hand and cluster of galaxies on the other) that differ by many orders of magnitude from the dimension of the physical bodies to which we adapted during our biological evolution. The difficulty then is as follows: if one believed in this sort of Kantian explanation, supplemented by whatever part of evolutionary psychology or cognitive psychology may come to help, one would have to say that it is only through analogy that we can extend what works at our dimension to other, much smaller or much bigger dimensions. But notoriously the analogies between the laws of the atom and those of classical physics break down, and we certainly have no direct experience of atoms and molecules. And yet mathematics applies successfully also to the quantum world, which in Kantian terms is a sort of noumenal world going beyond experience. It seems that the Kantian position needs to answer this objection, at least to the extent that it is plausible to assume, as it is, the mind-independent existence of atoms and molecules:³ if mathematics applies to the objects of our experience because the latter is possible *only via* the primitive mathematical notions that belong to our subjective side, how can we apply mathematics so successfully to objects that are *beyond* the reach of experience? Why should evolution have equipped us with the laws of objects that, like atoms, play no role in our ordinary life?

A possible reply to this objection is suggested by Steiner: scientists extend the applicability of mathematics by using Pythagorean or syntactic-formal *analogies* between physical laws written in purely mathematical language, for which no translation into non-mathematical language is possible. This suggestion can have a two-fold interpretation. Either scientists correctly believe that the world is objectively written in mathematical language, as Galilei thought, or the success of mathematics in extra-phenomenal realms is utterly unexplainable, because it is the result of pure chance, the coincidence being given, for instance, by the fact that physical

³ The way out of denying the existence of mind-independent entities is of no use, since the predictive power of structures like that of the Hilbert space is undeniable and needs to be explained.

laws suggested by mathematical analogies with laws working within the domain of macroscopic bodies work also within the quantum domain.

In either hypothesis, however, the Kantian position is in trouble, because it makes the explanation of the effectiveness of mathematics impossible.

3.3 Mathematics as a kind of language

The third position is very close to the second, and argues that mathematics is a type of language, so that the question of the applicability of mathematics is a chapter of a generalized theory of semantics. Ordinary language is successful in describing, predicting and explaining many properties of ordinary objects: mathematics is just a sophistication of these abilities, since relative to the natural languages, mathematics is just a more rigorous, less ambiguous, and formally organized language. Chomsky's generative linguistics and Fodor's computational view of human thought (Fodor, 1975) may give support to this third position, since besides the algorithmic nature, mathematics seems to share with both language and thought the characters of *productivity and systematicity*. The number of mathematical results and theorems that can be generated from certain premises seems potentially infinite, and mathematics keeps growing in many different directions simply by building in a recursive and combinatorial way on previous results (productivity). All branches of mathematics are deeply connected with each other, so that the capacity to generate a certain result is intrinsically connected to the capacity of generating other results (systematicity).

However, a linguistic approach to the problem of effectiveness of mathematics is affected by the difficulty of finding non-mathematical correlates for central mathematical concepts. This is certainly *not* the case with ordinary languages, whose referential terms have always a well-defined extension in the outer world. Consider the following list of notions, characterized by an increasing abstraction. We do not have too many troubles in finding a non-mathematical correlate for the mathematical notion of *subtracting* two numbers, since it corresponds to the physical operation of separating objects that were previously together; likewise, the mathematical notion of "*linearity*" corresponds to the superposition of two waves or to the fact that two causes contribute separately to their effect. *Fiber bundles* may be taken to describe gauge fields, but what does the *analyticity* of a function correspond to in the real world?⁴ While the latter mathematical notion is crucial in the applicability of many parts of mathematics, it does not seem to have any counterpart in the physical world. Once again, unless we assume

⁴ Some of these examples are due to Steiner (1998).

that any important mathematics notion can be parsed in a non-mathematical language, which seems difficult, we have no way of making sense of the applicability of the symbolic language of mathematics to the physical world. A literary quotation addressing the role of complex numbers, due to the German writer Robert Musil, will conclude this brief discussion of the third position (Musil, 1906):

The strange fact is that with these imaginary or even impossible numbers one can anyway make perfectly real calculations which end in a concrete result. (p. 56)

Ironically, at the time of *The Confusions of the Young Törless*, from which this passage is taken, Musil could not be aware at the fact that the most successful theory of the atomic structure of matter – quantum mechanics – would have been using complex, “imaginary” numbers to calculate the probability of measurements.

3.4 Mathematics as the science of the abbreviation of sequences

We can regard a mathematically formulated law as a “bridge” colligating two banks of a river, each constituted by quantitative data resulting from measurements. On one side of the river we find the initial data or boundary conditions, which in our metaphor we can regard as the *input*, and on the other side we find the predictions or retrodictions – the *output* – the result of a *calculus*.

Since such a result is obtained in a purely deductive fashion, that is, thanks to the application to the initial data of a mechanical rule given by the physical law, the metaphor of the scientific laws regarded as the algorithm of a computer appears initially justified. If the initial data in fact are such as to satisfy some mathematical conditions which in this context can be omitted,⁵ and whenever the solution to the equation exists and is unique, a mathematical law expressed as a differential equation enables us to transform in a finite numbers of steps, and in purely mechanical fashion, the initial data in final predictions or retrodictions (output).

What interests us is, of course, whether such an analogy between the laws of succession of any physical system – regarded as something that evolves in time by going through a finite number of states describable in physical language – and the software of a computer, can help us to better understand: (i) why the world is describable by mathematical laws and (ii) how the latter

⁵ The functions representing the data must be differentiable at least as many times as the degree of the differential equation giving the algorithm.

are related to the world, that is, how they *represent* it. In order to shed light on the presuppositions of the law-software analogy, we should ask whether also a physical system, in a sense to be specified, could be said to “compute” its “next” state *by bringing it about*. Let us assume, for the sake of the argument, that it makes literal sense to claim that a physical system going from an initial to a final state *literally executes a program or calculates its future state*, in the same sense in which a mathematical physicist deduces or calculates the predictions corresponding to the initial data.

We will now discuss more closely this view of the applicability of mathematics, in order to evaluate its chance to be a candidate to meet Wigner’s challenge.

4. THE ALGORITHMIC VIEW OF THE LAWS OF NATURE

Suppose that the temporal evolution of any physical system is describable by *finite strings of real numbers*, corresponding to operational measures of physical magnitudes (temperature, pressure etc.). We can have two cases: such strings can be ordered

(111000111000111000...)

or truly random

(0100110101100110...).

In the first case, the string can be generated by a simple instruction (“print 111000 n times”), which is much shorter than the list itself. In the second case, the string *appears* as truly random, where “appear” is meant to stress that while we can show that a finite string is not random by giving the generating law (algorithm), we can never prove that a string is random (this is a version of the halting theorem).

At this point we can give two definitions, based on algorithmic complexity theory, which will be relevant for our purpose:

Definition1: the complexity of a string is the length of the shortest algorithm capable of generating it

Definition2: a string is said to be algorithmically compressible when there is an algorithm capable of generating it, such that its information content (number of bits) is much less than that of the string.

As an illustration of these definitions, consider that a string like

{1, 4, 9, 16, 25, 36, ...} (1)

is obviously not random, since it can be trivially obtained by squaring the positive integers in the list

$$\{1, 2, 3, 4, 5, 6, \dots\} \quad (2)$$

If the numbers in (2) correspond to measured magnitudes in such a way that, say in a temporal interval of 1, 2, 3 seconds (the input data), a body travels 1, 4, 9 meters (the output), then the existence of a rule generating (1) from (2) shows that (1) *is algorithmically compressible*. The above algorithm is – *modulo* the constant $\frac{1}{2}g$. – Galileo’s law of free fall, generating the spatial intervals (1) from the square of the temporal intervals (2).

In a word, by following the metaphor of scientific laws regarded as the software of a physical system, we discover that *searching for laws is tantamount to asking which is the length of the shortest program capable of generating the string of numbers expressing the experimental measures*. Such a length – the complexity of the string – will be equal to that of the original string only if the latter is composed by apparently random numbers, and does not obey any known law.

The idea that scientific laws are an economic synthesis of all the information contained in our observations is certainly not new, and in this algorithmic approach it finds a new, rigorous and precise formulation. It was especially Ernst Mach who regarded science and its theories and laws as an economic “summary” of our observations. As he wrote (Mach, 1896):

Science is a form of business. Its purpose is to find the maximum amount of the infinite eternal truth with the minimum amount of work, in the minimum expenditure of time and with the minimum amount of thought effort. (p. 14)

After having made explicit the philosophical consequences that seem to follow from the software metaphor for scientific laws, we can now finally discuss a possible explanation of the applicability of mathematics, due to the physicist John Barrow (Barrow, 1992):

science exists because the natural world seems algorithmically compressible. The mathematical formulae that we call laws of nature are economical reductions of enormous sequences of data expressing changes of state of the world: here is what we mean by intelligibility of the world...Since *the physical world is algorithmically compressible*, mathematics is useful to describe it because it is the language of the abbreviation of sequences. The human mind enables us to make contact with that world because our brain has the ability of compressing complex sequences of sense data in shorter form. Such abbreviations make thought and memory possible. The natural limits that nature poses to our

senses prevent us from overloading our brains with information about the world. Such limits are security gates for our minds. (p. 93-96)

5. SOME DIFFICULTIES OF THE ALGORITHMIC VIEW

In discussing some of the difficulties generated by this position, let us start to discuss parts of Barrow's quotation.

(1) On the "epistemic" hand, if it is the brain that filters sense data and elaborates them through the construction of mathematical concepts, what is the relationship between such a capacity and the applicability of mathematics, regarded as the technique of compressing sequences? This is an open question and it connects this approach with the second (and third) position. Until we have a clear answer to this question, it is difficult to credit Barrow's claim for being more than an interesting speculation.

(2) On the "ontic" hand, it is not clear what it means to affirm that "the physical world is compressible": isn't this another way of formulating what we are trying to explain? Furthermore, note that compressibility is an *epistemic* notion: we are interested in compressing information, nature isn't. However, we are after an *ontological* interpretation of laws regarded as algorithms, enabling us to understand why mathematics is applicable with success.

(3) As persuasively shown by McAllister in a different context (McAllister, 2003), strings of data have interest-relative patterns. In his example, McAllister asks us to consider a very long string of data on the atmospheric temperature: we can find many regular patterns with different periods. We will find regularities with a period of a day (due to the rotation of the earth), other patterns lasting some days (due to the weather systems), one lasting a year (due to the earth's revolution around the sun), patterns which are repeated every 11 years (due to the sun spots), other regularities which are 21000 years long (due to the precession of the earth's orbit). Each of these patterns has a different algorithmic complexity. Which is the *intrinsic* algorithmic or effective complexity of the string of data? Since the answer to this question is interest-relative, this approach cannot be used for any ontological purpose.

(4) Not all laws of nature are sequential (i.e., laws of succession) as the notion of laws as algorithm requires! Since the notion of algorithm is essential temporal (even in parallel-distributed computation, the results of the distinct calculations must interact before the output), either all natural laws are laws of succession, or else natural laws cannot be given an

ontological interpretation within the algorithmic view, because *ontologically interpreted laws have to be executed by physical systems*. Tale laws such as

$$F = G(M_1 M_2)/r^2 \quad (3)$$

$$\Phi = e_0 q \quad (4)$$

$$PV = kT \quad (5)$$

Either they all derive from laws of succession, or physical systems cannot be said to instantiate them, because their parts would have to communicate at superluminal speed. A law of coexistence in fact links by definition parts of physical systems that are spacelike-related.

Before concluding, it is important to clarify what this objection does *not* entail. Clearly, we *can* use the laws in Eqs.(3)-(5) to calculate one side of the equation from the other, and in this sense their algorithmic character is obviously not refuted. However, calculations to humans take time, and note that – in order to give the current interpretation of laws an ontic significance – we were assuming that physical systems do go from one of their states to another one by executing an algorithm. It is in this sense that only temporally related states of a system can execute an algorithm and only laws of succession can be captured by the algorithmic view of the laws of nature.

6. CONCLUSION

As it should be obvious from the above survey, the problem of the effectiveness of mathematics is here to stay, and no one of the solutions that we have sketched here is devoid of serious difficulty. Clearly, research in this stimulating area calls for a multidisciplinary effort, coming from philosophers of mathematics, historians of philosophy, epistemologists, linguists, historians of science, cognitive scientists and possibly neuro-physiologists. And the problem is of immense interest also to physicists, as the contemporary Nobel prize Weinberg admits that “it is positively spooky that the physicist finds the mathematician has been there before him” (Weinberg, 1986, p. 725).

Finally, Wigner’s problem has all the trademark of a deep philosophical problem, since not only does it favour the dialogue between the science and the humanities, but also helps us to understand the place of mankind in the universe.

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