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DIPARTIMENTO DI ECONOMIA

A SOCIAL WELFARE FUNCTION CHARACTERIZING COMPETITIVE EQUILIBRIA OF INCOMPLETE FINANCIAL MARKETS

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Working Paper n° 93, 2008

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A social welfare function characterizing competitive equilibria of incomplete financial markets

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May 2008

Abstract

A classic characterization of competitive equilibria views them as feasible allocations maximizing a weighted sum of utilities. It has been applied to establish fundamental properties of the equilibrium notion, such as existence, determinacy, and computability. However, it fails for economies with missing financial markets.

We give such a characterization for economies with missing financial markets, by an amended social welfare function. Its parameters capture both the relative importance of households' welfare–the classic weights–as well as the disagreements among them as to the value of the missing markets.

As a by-product, we identify the dimension of the set of interior equilibrium allocations.

Keywords: incomplete markets, social welfare function, manifold

JEL Classification: D52, D61.

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1 Introduction

If no financial markets are missing, following Lange (1942) and Allais (1943), interior allocations of given resources are competitive equilibria if and only if they solve the program $\max_{\Sigma x^h = r} W_\delta(x)$ for some strictly positive δ, with W_δ being the social welfare function[1]

$$W_\delta(x) := \Sigma \delta^h u^h(x^h) \tag{1}$$

The parameters δ in Lange's social welfare function capture the relative importance of households' welfare. This characterization has been applied to establish fundamental properties of the equilibrium notion–existence, Negishi (1960) and Bewley (1969), determinacy with infinitely lived households, Kehoe and Levine (1985), and computability, Mantel (1971).

If some financial markets are missing as in Radner (1972), however, this equivalence fails: some interior competitive equilibria need not solve the program $\max_{\Sigma x^h = r} W_\delta(x)$ for any strictly positive δ. Moreover, no natural social welfare function W has been found that would rescue this implication.

We extend the characterization to economies with some missing financial markets, by amending the social welfare function. Thus interior allocations of given resources are competitive equilibria if and only if they solve the program $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ for some parameters $\delta \in \mathbb{D}, \mu \in \mathbb{M}$ living in certain spaces, with $W_{\delta,\mu}$ being the social welfare function

$$W(x) := \Sigma \delta^h u^h(x^h) - \Sigma \mu^h \cdot x^h_{\mathbf{1}} \tag{2}$$

Here, the *social evaluation* of allocations is described by the usual weights δ on households' welfare, and by new charges μ on their future consumption. The parameter δ is interpreted classically, whereas μ is interpreted as the "disagreement" among households as to the "value" of the "missing financial markets," as justified below.

Why does it fail, the equivalence of competitive equilibria and maxima of (1), if some financial markets are missing? On the one hand, any allocation x that maximizes this is Pareto efficient. Indeed, if y were Pareto superior to x, i.e. $(u^h(y^h)) > (u^h(x^h))$, then $W_\delta(y) > W_\delta(x)$ for any $\delta \gg 0$, so x could not be a maximum for any $\delta \gg 0$. On the other hand, some allocations x that are competitive equilibria of incomplete financial markets are Pareto inefficient. Indeed, for almost every initial allocation, every competitive equilibrium allocation is Pareto inefficient—for an exposition of this well known fact, see Magill and Quinzii (1996).[2] So some competitive equilibria fail to maximize (1) for any $\delta \gg 0$.

We explain in what sense the parameter μ is the "disagreement" among households as to the "value" of the "missing financial markets," by clarifying each of these terms. By "missing financial markets" we mean the orthogonal complement $a^\perp$ of the span of the existing financial instruments a. By "value" of the missing financial markets we mean a linear functional $v : a^\perp \to \mathbb{R}$. The Riesz representation theorem

[1] Lange characterizes *Pareto optima* in this way. So the above characterization follows from the two welfare theorems. (Lange (1942) is aware of the first one, while Allais (1953) is among the first to rigourously prove the second one.)

[2] If there are multiple goods and enough missing financial markets, even the equilibrium use of the existing financial markets is generically Pareto inefficient, as shown by Geanakoplos and Polemarchakis (1986), who pioneer the application of transversality to equilibrium welfare. The intuition for this is due to Stiglitz (1982). A sweeping generalization is in Citanna, Kajii and Villanacci (1998).

or separating hyperplane theorem imply that any linear functional on a finite dimensional inner product space can be represented uniquely as the inner product against a unique element of the vector space–call this element $\hat{v} \in a^{\perp}$, so that $v(m) = m \cdot \hat{v}$. If each household thinks such a *value* v^h, the *disagreement* is then the differences from the mean, $\mu^h := \hat{v}^h - mean(\hat{v}^1, ..., \hat{v}^H)$. When so defined, the disagreement $\mu = (\mu^h)$ satisfies two properties: (i) $\mu^h \in a^{\perp}$, because it is a linear combination of points $\hat{v}^h \in a^{\perp}$ in a vector space, and (ii) $\Sigma \mu^h = 0$, because these are differences from the mean. In sum, imagining that each household has its own v^h, an opinion as to the value of the missing financial markets, then this is the sense of the new parameter in our social welfare function (1)—a matrix $\mu = (\mu^h)$ satisfying conditions (i), (ii).

Our main conclusions are about the following set, given some smooth preferences u ‘a la Debreu (1972), some state-contingent resources r, and some finitely many financial instruments a. Namely, the set $\mathbb{X}$ of all interior competitive equilibrium allocations arising from some income distribution $\Sigma e^h = r$ compatible with the resources.

The first result (theorem 1) is that an allocation $x \gg 0$ is an equilibrium allocation if and only if it solves the program $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ for some $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$, where

$$\mathbb{D} := \left\{ \delta \in \mathbb{R}^H \mid \delta \gg 0, \Sigma \frac{1}{\delta^h} = 1 \right\} \qquad \mathbb{M} := \left\{ \mu \in \left(a^{\perp}\right)^H \mid \Sigma \mu^h = 0 \right\}$$

We see that the "welfare" parameter δ is normalized in a standard way, and the "disagreement" parameter μ reflects properties (i) and (ii) above.

The second result (proposition 1, part A) identifies the (δ, μ) from the equilibrium allocation as being

$$\begin{aligned} \delta^h(x) &= \frac{1}{D_{x_0} u^h(x^h)} \\ \mu^h(x) &= \hat{v}^h - mean(\hat{v}^1, ..., \hat{v}^H) \text{ with } \hat{v}^h := \frac{D_{x_1} u^h(x^h)}{D_{x_0} u^h(x^h)} \end{aligned} \tag{3}$$

Thus δ^h is the inverse of the marginal utility of present consumption, as usual, and μ is, as interpreted above, the disagreement among households as to the value of the missing financial markets, where each household's "value" $\hat{v}^h$ is concretized as the marginal rates at which it substitutes consumption in future states for consumption in the present state. Here, the abstract notion of "value" as a linear functional $v : a^{\perp} \to \mathbb{R}$ is made concrete by the idea of marginal willingness to pay as $\Delta \mapsto \Delta \cdot MRS$, the inner product of the infinitesimal change Δ in future consumption against the marginal rates of substitution MRS.

The third result (theorem 2) is that the relation $x \leftrightarrow (\delta, \mu)$ between $\mathbb{X} \leftrightarrow \mathbb{D} \times \mathbb{M}$ is a bijection, smooth in both directions. This implies immediately that the dimension of $\mathbb{X}$ equals the dimension of $\mathbb{D} \times \mathbb{M}$, which is easily shown to be $(H-1)(1+m)$ where m is the number of missing financial markets. This nests a well known fact about complete markets, where $m = 0$: the interior Pareto optima (which are $\mathbb{X}$ by the two welfare theorems) have dimension $H - 1$, cf. proof of 5.2.4 in Balasko (1988).

We restrict attention to an exchange economy that, for simplicity, has a single good per state and assets paying off in terms of it. However, our results extend to the case of multiple goods per state and any assets paying off linearly in the goods' prices, with the social welfare function and arguments being almost identical. For this more general setting, Tirelli (2008) develops a parameterization of equilibria, alternate to the $x \leftrightarrow (\delta, \mu)$ here, emphasizing their geometry over the social welfare function they optimize. He then applies the parameterization to derive the constrained inefficiency of equilibria in the sense of Geanakoplos and Polemarchakis (1986).

By way of application, we suggest the problem of computing equilibria of incomplete financial markets. This problem has its analogue with complete financial markets, for which many algorithms and their convergence properties are available. One such algorithm is Mantel's (1971), a dynamic system in the welfare weights δ given initial endowments e :

$$\dot{\delta}^h = \frac{p_\delta'(e^h - x_\delta^h)}{\delta^h}$$

where x_δ solves the δ-program $\max_{\Sigma x^h = r} W_\delta(x)$ and p_δ is its Kuhn-Tucker multiplier, with W_δ as in (1). Clearly, a rest point $\dot{\delta} = 0$ corresponds to an allocation x_δ that is resource-feasible, and Marshallian-optimal relative to prices p_δ, so that $x_\delta \in \mathbb{X}$. He shows that if the utilities define excess demands for which goods are gross-substitutes—known to imply unique equilibrium prices—and are homothetic, then this dynamic system is globally stable. Our characterization suggests that a natural idea for computing equilibria of incomplete financial markets a, would be to amend his dynamic system to one in the parameters δ, μ, and amend his condition for global stability.

The paper proceed as follows. Section 2 spells out the model and assumptions. Section 3 develops the characterization. Section 4 refines the characterization, computing the dimension of $\mathbb{X}$. Section 5 contains the more formalistic and less insightful proofs.

2 Economy and equilibria

Primitives There are $h = 1, ..., H$ *households* who know the *present state of nature* 0 but are uncertain as to which *future state of nature* $s = 1, ..., S$ will occur. In each state a nonstorable *good* is available for *consumption*, and in state 0 there are *assets* $j = 1, ..., J$ available for *trade*.

Real economy The *resource* $r \in \mathbb{R}^{S+1}_{++}$ of the good is state-contingent, and the *income distribution* e across households is compatible, $\Omega := \left\{ e \in \mathbb{R}^{H(S+1)}_{++} : \Sigma e^h = r \right\}$. Each asset j pays off in the future a state-contingent amount $a^j \in \mathbb{R}^S$ of the good, summarized by a matrix $a \in \mathbb{R}^{S \times J}$.[3] Asset markets are **complete** if $span(a) = \mathbb{R}^S$, **incomplete** otherwise.

Markets Markets specify that each asset j is tradeable at a price of q^j units of the good in the present, by specifying $q = p'a$ (row) for some *state prices* $p \in \mathbb{R}^S_{++}$. $Q \subset \mathbb{R}^J$ denotes such asset prices. Households are free to trade any amount $\theta^h_j \in \mathbb{R}$ of any asset: buy $\theta^h_j > 0$, sell $\theta^h_j < 0$, or neither $\theta^h_j = 0$. Trades of asset j **clear** if $\Sigma \theta^h_j = 0$. Viewing asset prices as a negative payoff in the present, asset payoffs become $W := \begin{pmatrix} -q \\ a \end{pmatrix} \in \mathbb{R}^{S+1 \times J}$

Remark 1 *The payoffs a of the assets and the resources r of the good are fixed throughout the paper. This is important in interpreting the dimensions reported in section 4.*

[3] All vectors are column vectors, unless stated otherwise.

Consumption The **consumption correspondence** from asset prices $q \in Q$ and one's income $e^h \in \mathbb{R}^{S+1}_{++}$ is

$$X(q, e^h) = span\left(e^h + W\mathbb{R}^J\right) \cap \mathbb{R}^{S+1}_{+}$$

The **asset trade** is any function $\theta : Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^J$ from asset prices and one's income. Each household has an asset trade θ^h, by which it transforms its income into consumption, $x^h(q, e^h) := e^h + W\ \theta^h(q, e^h)$. Trades are **optimal** if there is a utility function $u^h : \mathbb{R}^{S+1}_{+} \to \mathbb{R}$ solving

$$u^h(x^h(q, e^h)) = \sup u^h(X(q, e^h))$$

Definition 1 *$(q, e) \in Q \times \Omega$ is an **equilibrium** if asset trades clear, $\Sigma\theta^h(q, e^h) = 0$. It is a **no-trade equilibrium** if $\theta^h(q, e^h) = 0$ for every h.*

We denote by $\mathbb{E}, \mathbb{T}, \mathbb{X}$ the sets of equilibria, no-trade equilibria, equilibrium allocations–an **equilibrium allocation** is any x for which $(q, x) \in \mathbb{T}$ for some $q \in Q$.

2.1 Assumptions

Assumption 1 *In the economy, the income distribution is strictly positive* ($e \in \Omega$) *and no assets are redundant* (a *has rank* J).

Assumption 2 *Trades by h are optimal with respect to utility u^h.*

Assumption 3 *u^h is continuous, C^2 in $\mathbb{R}^{S+1}_{++}$, strictly increasing ($\forall x \in \mathbb{R}^{S+1}_{++}, Du^h(x) \gg 0$), strictly concave ($\forall x \in \mathbb{R}^{S+1}_{++}, D^2u^h(x)$ is negative definite), and boundary averse ($\forall x' \in \mathbb{R}^{S+1}_{++}, u^h(x) \geq u^h(x') \Rightarrow x \in \mathbb{R}^{S+1}_{++}$, and $lim_{x_s \searrow 0} \frac{\partial u^h(x)}{\partial x_s} = \infty$).*

An instrumental notion is each household's ∇^h (**marginal rates of substitution**), the row-vector function $\mathbb{R}^{S+1}_{++} \to \mathbb{R}^{S+1}_{++}$

$$\nabla^h(x) := \left(.., \frac{D_{x_s}u^h(x)}{D_{x_0}u^h(x)}, ..\right) \tag{4}$$

It is instrumental because of a well known implication of the assumptions, that the optimal asset trades $\theta^h(q, e^h)$ are C^1 and characterized as the unique solution of

$$\nabla^h a - q = 0 \tag{θ}$$

while evaluating ∇^h at $e^h + W\theta^h$.

3 Equilibrium allocations characterized

We characterize equilibria as solutions of the program $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ for some parametric social welfare function $W_{\delta,\mu}$, where the parameters satisfy a specific restriction, $(\delta, \mu) \in \mathbb{P}$. The social welfare function in question, given parameters $(\delta, \mu) \in \mathbb{R}^H \times \mathbb{R}^{HS}$, is

$$W_{\delta,\mu}(x) := \Sigma\delta^h u^h(x^h) - \Sigma\overline{\mu}^h \cdot x^h \tag{5}$$

where $\overline{\mu}^h = (0, \mu^h) \in \mathbb{R}^{S+1}$ simply prepends to the row vector μ^h a coordinate with value zero.

Key to the characterization is a function $(\delta(x), \mu(x))$ from income distributions Ω to the ambient space $\mathbb{R}^H \times \mathbb{R}^{HS}$, given by

$$\delta^h(x) = \frac{t}{D_0 u^h} \qquad \mu^h(x) = t(\nabla^h - \overline{\nabla}) \tag{6}$$

where

$$t = \Sigma D_0 u^i \in \mathbb{R}_{++} \qquad \overline{\nabla} = \frac{1}{H}\Sigma\nabla^h \in \mathbb{R}^S_{++} \tag{7}$$

by assumption 3. Since the dependence of ∇^h on x^h, and of $t, \overline{\nabla}$ on x is understood, it is being omitted.

In a nutshell, the logic of the characterization has two steps. In the "necessity step" (proposition 1), we show that if $x \gg 0$ is an equilibrium allocation, then it solves the program for $(\delta, \mu) = (\delta(x), \mu(x))$, the value of (6) at the equilibrium allocation, and in turn $(\delta(x), \mu(x))$ satisfies specific restrictions. It is then natural to define, independently of the social welfare function (5) or the function (6), the subset $\mathbb{P} \subset \mathbb{R}^H \times \mathbb{R}^{HS}$ of all points satisfying these restrictions. In the "sufficiency step" (proposition 2), we show that an allocation x that solves the program for some $(\delta, \mu) \in \mathbb{P}$ is an equilibrium allocation, and in turn (δ, μ) must be $(\delta(x), \mu(x))$, the value of (6) at the maximum. We include all proofs in the body of the paper because they are insightful and simple.

Proposition 1 (necessity) *If $x \gg 0$ is an equilibrium allocation, then*

- (A) it solves the program $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ for the $(\delta, \mu) = (\delta(x), \mu(x))$ in (6)
- (B) $(\delta(x), \mu(x))$ satisfies the restrictions $\delta(x) \gg 0, \Sigma\frac{1}{\delta^h(x)} = 1$ and $\mu^h(x) \in a^\perp, \Sigma\mu^h(x) = 0$.

Proof. Part A Feasibility: $\Sigma x^h = r$ holds because x is an equilibrium allocation. Maximality: By the easy half of Kuhn-Tucker, it suffices that x maximizes

$$W_{\delta,\mu}(x) - \rho \cdot \Sigma x^h$$

for some $\rho \in \mathbb{R}^{S+1}$, say, $\rho := t(1, \overline{\nabla}) \gg 0$ with $t, \overline{\nabla}$ as in (7).

Note $W_{\delta,\mu}(x) - t(1, \overline{\nabla}) \cdot \Sigma x^h$ is concave, given $\delta = \delta(x) \gg 0$. So it is maximized at $x \gg 0$, the equilibrium allocation, so long as its derivative is zero there. Its derivative with respect to x^h_0 is

$$\delta^h D_0 u^h - t$$

By the hypothesis that δ is the $\delta(x)$ in (6), this is zero indeed. Its derivative with respect to $x^h_{\mathbf{1}}$ where $\mathbf{1} = \{1, ..., S\}$ is

$$\delta^h D_{\mathbf{1}} u^h - \mu^h - t\overline{\nabla}$$

By the hypothesis that δ, μ are the $\delta(x), \mu(x)$ in (6), this is

$$\begin{aligned} & \frac{t}{D_0 u^h} D_{\mathbf{1}} u^h - t(\nabla^h - \overline{\nabla}) - t\overline{\nabla} \\ = \quad & t\frac{D_{\mathbf{1}} u^h}{D_0 u^h} - t\nabla^h \end{aligned}$$

Recalling definition (4), this is zero indeed.

Part B | $\delta(x) \in \mathbb{D}$: Definitions (6), (7) immediately imply the $\frac{1}{\delta^h} = \frac{D_0 u^h}{t}$ sum to 1; also, $\delta^h > 0$ by assumption 3; therefore, $\delta(x) \in \mathbb{D}$. $\mu(x) \in \mathbb{M}$: It suffices that $(\nabla^h - \overline{\nabla})_h \in \mathbb{M}$, since $t = \Sigma D_0 u^i > 0$ is just a uniform rescaling. That the $(\nabla^h - \overline{\nabla})$ sum to 0 is immediate from the definition of $\overline{\nabla} := \frac{1}{H}\Sigma\nabla^h$ as the average. (Up to here, the proof does not require the hypothesis of equilibrium.) That $\nabla^h - \overline{\nabla} \in a^\perp$ follows from asset trades, which are optimal in equilibrium hence satisfy $\nabla^h a - q = 0$ as noted in (θ). Averaging these equations, $\overline{\nabla} a - q = 0$. Subtracting the latter from the former, $(\nabla^h - \overline{\nabla})a = 0$, i.e. $\nabla^h - \overline{\nabla} \in a^\perp$. ■

Let us define the sets

$$\mathbb{D} := \left\{\delta \in \mathbb{R}^H \mid \delta \gg 0, \Sigma\frac{1}{\delta^h} = 1\right\} \qquad \mathbb{M} := \left\{\mu \in \left(a^\perp\right)^H \mid \Sigma\mu^h = 0\right\} \tag{8}$$

Whereas these sets are defined independently of the auxiliary function (6), conclusion (B) does refer to the auxiliary function (6). Yet it is possible to paraphrase conclusion (B) in terms of these sets: $(\delta(x), \mu(x)) \in \mathbb{D} \times \mathbb{M}$. Putting together (A) and (B) therefore yields the

Corollary 1 *If* $x \gg 0$ *is an equilibrium allocation, then it solves the program* $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ *for some* $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$.[4]

A natural conjecture is whether the converse is true, and it is:

Proposition 2 (sufficiency) *If* $x \geq 0$ *solves the program* $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ *for some* $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$, *then*

- (A) it is an equilibrium allocation
- (B) (δ, μ) is necessarily the $(\delta(x), \mu(x))$ in (6)—in particular, $x \gg 0$.

Proof. Part B That $x \in argmax \gg 0$ follows from the boundary aversion in assumption 3 and $\delta \gg 0$. Further, the harder half of Kuhn-Tucker implies that x maximizes

$$W_{\delta,\mu}(x) - \rho \cdot \Sigma x^h$$

for some $\rho_+ = (\rho_0, \rho) \in \mathbb{R}^{S+1}$. (Here we use that $W_{\delta,\mu}$ is concave and $\Sigma x^h = r$ linear, so that the constraint qualification automatically holds.) Since $x \gg 0$, the derivative must be zero:

$$\delta^h D u^h - \overline{\mu}^h = \rho_+ \tag{9}$$

Equation (9) implies for state 0 that

$$\delta^h = \frac{\rho_0}{D_0 u^h}$$

since $\overline{\mu}^h_0 = 0$. This and the hypothesis $\delta \in \mathbb{D}$ (so that $\Sigma\frac{1}{\delta^h} = 1$) imply $\rho_0 = \Sigma D_0 u^i$, hence

$$\delta^h = \frac{\Sigma D_0 u^i}{D_0 u^h} \tag{10}$$

[4]The $\mathbb{P}$ to which the introduction to this section alludes is here recognized as $\mathbb{P} = \mathbb{D} \times \mathbb{M}$.

This states δ is the $\delta(x)$ in (6). And equation (12) implies for states $\mathbf{1} = \{1, ..., S\}$ that

$$\mu^h = \delta^h D_{\mathbf{1}} u^h - \rho = (\Sigma D_0 u^i) \nabla^h - \rho$$

on substituting conclusion (10) and definition (4). This and the hypothesis $\mu \in \mathbb{M}$ (so that $\Sigma \mu^h = 0$), when averaged, imply $0 = \frac{1}{H} \Sigma \mu^h = (\Sigma D_0 u^i) \overline{\nabla} - \rho$ hence $\rho = (\Sigma D_0 u^i) \overline{\nabla}$, which substituted back implies

$$\mu^h = (\Sigma D_0 u^i)(\nabla^h - \overline{\nabla}) \tag{11}$$

This states μ is the $\mu(x)$ in (6).

Part A By definition of equilibrium allocation, we are to show that $(q, x) \in \mathbb{T}$ is a no-trade equilibrium, i.e. $\theta^h(q, x^h) = 0$, for some asset prices $q \in Q$. By (θ), this is equivalent to $\nabla^h a - q = 0$ while evaluating ∇^h at the $x^h + W0 = x^h$, for some $q \in Q$. It suffices that this be true for, say, $q := \overline{\nabla} a$ with $\overline{\nabla}$ as in (7). That is, it suffices that

$$\nabla^h a - \overline{\nabla} a = 0 \tag{12}$$

while evaluating at x^h. Now, (12) is equivalent to $(\nabla^h - \overline{\nabla}) a = 0$ is equivalent to $(\nabla^h - \overline{\nabla}) \in a^\perp$ is equivalent to $(\Sigma D_0 u^i)(\nabla^h - \overline{\nabla}) \in a^\perp$ (since the vector space $a^\perp$ is closed under rescalings by $(\Sigma D_0 u^i) \neq 0$). Thus it suffices that $(\Sigma D_0 u^i)(\nabla^h - \overline{\nabla}) \in a^\perp$ while evaluating at x^h. This is true, because of conclusion (11) and the hypothesis $\mu \in \mathbb{M}$ (so that $\mu^h \in a^\perp$). ■

Putting together part (A) of propositions 1 and 2 yields our characterization of equilibria:[5]

Theorem 1 *Suppose* $x \gg 0$. *Then it is an equilibrium allocation iff it solves* $\max_{\Sigma x^h = r} W_{\delta,\mu}(x)$ *for some* $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$.

We remark that if asset markets are complete, then $a^\perp = \{0\}$ and $\mathbb{M} = \{0\}$ and $\mu = 0$ necessarily, making $W_{\delta,\mu}(x) = \Sigma \delta^h u^h(x^h)$. In particular, if asset markets are complete, theorem 1 simply concludes that x is an equilibrium allocation if and only if it solves $\max_{\Sigma x^h = r} \Sigma \delta^h u^h(x^h)$ for some $\delta \in \mathbb{D}$—the classical characterization.

As an aside, there is a separate characterization, which does not even refer to social welfare functions. The proof is relegated to the appendix, and simply recycles the arguments above.

Corollary 2 *Suppose* $x \in \Omega$. *Then it is an equilibrium allocation iff* $(\nabla^h - \overline{\nabla}) \mid_x \in a^\perp$ *for every* h.

Likewise, if asset markets are complete, then $a^\perp = \{0\}$ and this merely concludes that x is an equilibrium allocation if and only if it all ∇^h are equal (to the average)—a classical characterization.

Remark 2 (multiple goods) *If there are multiple goods per state and assets pay off in the numéraire, theorem 1 holds exactly as stated, so long as the social welfare function is amended as follows:*

$$\Sigma \delta^h u^h(x^h) - \Sigma \overline{\mu}^h \cdot x^h_{\textit{numéraire in future}}$$

[5] Part (B) of proposition 1 may seem superfluous, but in fact was the *identifier* of $(\delta, \mu) \in \mathbb{D} \times \mathbb{M}$ as this theorem's necessary and sufficient condition. Part (B) of proposition 2 will play a role in the next section.

4 Dimension of equilibrium allocations

Here, we state a version of the characterization that is stronger by the fact it claims the above relation $x \leftrightarrow (\delta, \mu)$ is a **diffeomorphism**–a bijection, smooth in both directions. If the utilities are time separable $u(x) = u_0(x_0) + u_{\mathbf{1}}(x_{\mathbf{1}})$, and we think of the inverse of (6), we see δ determines the distribution of present consumption x_0, and, given δ, μ determines the distribution of future consumption $x_{\mathbf{1}}$.

By way of caveat, the dimensions reported here are to be interpreted for fixed asset payoffs a and resources r.

Theorem 2 *The no-trade equilibria* $\mathbb{T}$ *are diffeomorphic to* $\mathbb{D} \times \mathbb{M}$.[6] *In fact, a global chart is (6). Thus* $\mathbb{T}$ *is a smooth manifold of dimension* $(H-1)(S-J+1)$.

To see why this is the dimension, note that $\mathbb{T}, \mathbb{D} \times \mathbb{M}$ must have equal dimension, being diffeomorphic, while that of $\mathbb{D} \times \mathbb{M}$ is easy to compute. It is $\dim(\mathbb{D}) + \dim(\mathbb{M})$; clearly, glancing at (8) we see $\dim(\mathbb{D}) = H-1$; and $\dim(\mathbb{M}) = (H-1)(S-J)$, because choosing $\mu^{1,\dots,H-1}$ from $a^{\perp}$ (itself of dimension $S-J$) uniquely determines $\mu^H \in a^{\perp}$ as $\mu^H = -\Sigma_{h<H}\mu^h$. Thus

$$\dim \mathbb{T} = \dim \mathbb{D} + \dim \mathbb{M} = (H-1) + (H-1)(S-J) = (H-1)(S-J+1)$$

Theorem 2 nests a well known fact about complete markets, where $S = J:$ the interior Pareto optima (which are $\mathbb{X}$ by the two welfare theorems) have dimension $H-1$, cf. proof of 5.2.4 in Balasko (1988).

Corollary 3 *The equilibria* $\mathbb{E}$ *are a* $(H-1)J-$*vector bundle on* $\mathbb{T}$, *hence a smooth* $(H-1)(S+1)$-*manifold.*

To see why this is the dimension, note that $\mathbb{E}$, as locally the Cartesian product of $\mathbb{T}$ and a vector space of dimension $(H-1)J$, must have dimension

$$\dim \mathbb{E} = \dim \mathbb{T} + (H-1)J =_{\text{theorem 2}}= (H-1)(S+1)$$

Corollary 3 agrees with a well known fact about complete markets, where $S = J:$ the equilibrium manifold given fixed resources has dimension $(H-1)(\#goods)$, cf. chapter 5 in Balasko (1988).

Remark 3 (multiple goods) *If there are multiple goods per state and assets pay off in the numéraire, theorem 2 holds exactly as stated–so the dimension of* $\mathbb{T}$ *stays the same.*

5 Remaining proofs

Starting from the well known fact[7] that

Proposition 3 $\mathbb{E}$ *is a smooth manifold.*

our argument applies the very useful

[6] Note, though the statement is about $\mathbb{T}$ instead of $\mathbb{X}$, it this equivalent, as shown by the inverses $(q, x) \to x, x \to (\overline{\nabla}, x)$.
[7] Geanakoplos and Polemarchakis (1986), section 6.

Lemma 1 (3.2.1 in Balasko (1988)) *Let* $\phi : X \to Y, \psi : Y \to X$ *be smooth maps between smooth manifolds making* $\phi \circ \psi$ *the identity. Then* $\psi(Y)$ *is a smooth submanifold of* X *diffeomorphic to* Y*, and* $\phi\mid_{\psi(Y)}: \psi(Y) \to Y$ *is a diffeomorphism.*

where

$$X \text{ is } \mathbb{E}$$
$$Y \text{ is } \mathbb{D} \times \mathbb{M}$$

The maps are the following. $\phi : \mathbb{E} \to \mathbb{D} \times \mathbb{M}$ is

$$\phi(q,e) = \begin{bmatrix} \ldots, \frac{\Sigma D_0 u^i}{D_0 u^h}, \ldots \\ \ldots, (\Sigma D_0 u^i)(\nabla^h - \overline{\nabla}), \ldots \end{bmatrix} \tag{13}$$

evaluated at consumptions $e + W\theta\,(q,e)$. $\psi : \mathbb{D} \times \mathbb{M} \to \mathbb{E}$ is

$$\psi(\delta, \mu) = \left(\overline{\nabla} a, x\right) \tag{14}$$

where $\overline{\nabla} := \frac{1}{H}\Sigma\nabla^h$ is the average of the $\nabla^h(x)$ in (4) and

$$x := \arg\max_{x \in \overline{\Omega}} \Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h) \tag{15}$$

and $\overline{\mu}^h := (0, \mu^h)$.[8]

Lemma 2 ϕ *and* ψ *are well defined and satisfy the hypothesis in lemma 1.*

Proof. **of lemma 2.** $\boxed{\phi \text{ is well-defined}}$, i.e. $\phi(q,e) \in \mathbb{D} \times \mathbb{M}$. The proof of part (B) of proposition 1 applies verbatim.

$\boxed{\psi \text{ is well-defined}}$, i.e. $\psi(\delta, \mu)$ exists, is unique, and in $\mathbb{E}$. The argmax x exists because the objective in (15) is continuous and $\overline{\Omega}$ compact. The argmax x is unique because, first, $\arg\max \gg 0$ from the Inada condition in assumption 3, and, second, the objective is strictly concave in the interior from the concavity condition in assumption 3. It remains to show $(\overline{\nabla} a, x) \in \mathbb{E}$. We show this by showing $(\overline{\nabla} a, x) \in \mathbb{T}$, i.e. $\theta^h(\overline{\nabla} a, x^h) = 0$. The proof of part (A) of proposition 2 applies verbatim.

$\boxed{\phi \circ \psi = id}$ Given (δ, μ), define $(q, x) := \psi(\delta, \mu)$; we want $\phi\,(q,x) = (\delta, \mu)$. By part (B) of proposition 2, (δ, μ) equals the right side of (13) evaluated at allocation $x = \arg\max = \psi_2$. Also, the right side of (13) evaluated at allocation $x + W\theta(q,x)$, is the definition of $\phi(q,x)$, for any $(q,x) \in \mathbb{E}$. So the right sides agree, and therefore $(\delta, \mu) = \phi(q,x)$, if both allocations agree, i.e. if asset trades $\theta(q,x) = 0$. Now, $\theta(q,x) = 0$ holds because both $\theta(\overline{\nabla} a, \arg\max) = 0$—as shown in the proof of part (A) of proposition 2—and $(q,x) = (\overline{\nabla} a, \arg\max)$—because $(q,x) =_{hypothesis} \psi(\delta,\mu) =_{definition} (\overline{\nabla} a, \arg\max)$.

$\boxed{\text{Smoothness}}$ That ϕ is C^1 follows from its definition and that u^h is C^2, θ^h is C^1. That ψ is C^1 follows from the implicit function theorem by a standard argument that we omit. ■

Now the

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[8] $\overline{\Omega}$ is the closure of Ω.

Proof. of theorem 2. Lemma 2 verifies the hypothesis of lemma 1, so we deduce $\psi(\mathbb{D}\times\mathbb{M})$ is a smooth manifold diffeomorphic to $\mathbb{D}\times\mathbb{M}$, and $\phi\mid_{\psi(\mathbb{D}\times\mathbb{M})}: \psi(\mathbb{D}\times\mathbb{M})\rightarrow\mathbb{D}\times\mathbb{M}$ is a diffeomorphism. *Suppose for a moment that* $\psi(\mathbb{D}\times\mathbb{M})=\mathbb{T}$. Then we have deduced: $\mathbb{T}$ is a smooth manifold diffeomorphic to $\mathbb{D}\times\mathbb{M}$, and $\phi\mid_{\mathbb{T}}: \mathbb{T}\rightarrow\mathbb{D}\times\mathbb{M}$ is a diffeomorphism, i.e. a global chart, completing the proof of theorem 2. (For the proof of the statement about $\mathbb{T}'s$ dimension, see the paragraph just after the statement of theorem 2.)

We justify our momentary supposition that $\psi(\mathbb{D}\times\mathbb{M})=\mathbb{T}$.

The proof of lemma 2 shows $\psi(\mathbb{D}\times\mathbb{M})\subset\mathbb{T}$ (where ψ is shown well-defined), so we show $\mathbb{T}\subset\psi(\mathbb{D}\times\mathbb{M})$, by showing $id_{\mathbb{T}}=\psi\circ\phi\mid_{\mathbb{T}}$. Fix $(q,e)\in\mathbb{T}$. Write $(\delta,\mu):=\phi(q,e)$. We want $\psi(\delta,\mu)=(q,e)$. That is, what we want (by definition of ψ) is (i) $q=\overline{\nabla}\mid_e a$ and (ii) $e=\arg\max_{x\in\overline{\Omega}}\Sigma(\delta^h u^h(x^h)-\overline{\mu}^h x^h)$.

We show (i). We know equation (θ) holds at any equilibrium allocation. Since $(q,e)\in\mathbb{T}$, e is an equilibrium allocation. Therefore equation (θ) holds at $e: q=\nabla^h\mid_{e^h} a$. Averaged, it implies $q=\overline{\nabla}\mid_e a$.

We show (ii). Since $(q,e)\in\mathbb{T}$, e is an equilibrium allocation, so by part (A) of proposition 1, $e=\arg\max_{x\in\overline{\Omega}} W_{\delta,\mu}(x)$ for $(\delta,\mu):=\phi(q,e)$. ■

Finally, we provide the

Proof. of corollary 3. The projection $\mathbb{E}\rightarrow\mathbb{T},\pi(q,e)=(q,x(q,e))$ is well defined; its fibers $\pi^{-1}(q,x)$ are clearly

$$\pi^{-1}(q,x)=\{q\}\times\{e\in\Omega:\forall h, e^h=x^h-\Delta^h \text{ for some } \Delta^h\in span(W)\}$$

So fibers are parameterized by an open set of $\Delta^{h>1}$ in $span(W)^{H-1}$—here $e^1=x^1+\Sigma_{h>1}\Delta^h$—which is a convex set of dimension $(H-1)J$, depending smoothly on (q,e). ■

Finally, the

Proof. of corollary 2. Suppose $x\in\Omega$, hence $x\gg 0$. If it is an equilibrium allocation, then all $(\nabla^h-\overline{\nabla})\mid_x\in a^{\perp}$ according to part (B) of proposition 1. Conversely, if all $(\nabla^h-\overline{\nabla})\mid_x\in a^{\perp}$, we want that x is an equilibrium allocation, which by definition means that $(q,x)\in\mathbb{T}$, i.e. $\theta^h(q,x^h)=0$, for some asset prices $q\in Q$. The choice $q:=\overline{\nabla}a$ works, and the argument for why it works, coincidentally, is exactly the proof of part (A) of proposition 2. ■

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